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Jerzy Makowski
Helmut Stumpf

Finite strain theory of rods

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J. MAKOWSKI AND H. STUMPF

HIGHER ORDER FINITE STRAIN THEORY OF RODS

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Zusammenfassung

Für die Verformung ebener Biegungsträger unter Berücksichtigung großer Dehnungen wird eine neue Theorie vorgestellt. Das zugrunde liegende kinematische Modell berücksichtigt Längsdehnung, Biegung sowie Schubverformung des Trägers. Die hergeleiteten Feldgleichungen und Randbedingungen besitzen eine mathematische Struktur, die als ein-dimensionales Gegenstück zu der sogenannten nichteinfachen Theorie der Kontinuumsmechanik betrachtet werden kann. Zum Zwecke der Illustration wird die dargestellte Theorie zur Beulberechnung gerader Biegungsträger aus gummiartigem Material herangezogen.

Summary

A new theory is presented for finite strain deformation of planar rods. The adapted kinematical model is rich enough to accommodate extension, flexure, transverse shear and transverse normal shear of the rod. The field equations and boundary conditions derived here enjoy the mathematical structure that may be viewed as the one-dimensional counterpart of a non-simple theory known in continuum mechanics. For purpose of illustration the presented theory is applied to the buckling analysis of straight rods made of rubber-like material.

Contents

1. Introduction

In [18] we have formulated a fairly general theory of finite strain deformation of shells made of an incompressible hyperelastic material. The basic equations of this theory were derived from three-dimensional finite elasticity under the single assumption: material fibres initially normal to the shell reference surface remain straight in the process of an isochoric deformation. The resulting two-dimensional model for the shell is rich enough to accommodate not only flexure, extension and shear but also higher order effects incorporated in the dependency of the two-dimensional constitutive equations on the gradient of suitable strain measures. In this sense the shell theory developed in [18] (cf. also [23]) may be viewed as the two-dimensional counterpart of the non-simple material in continuum mechanics (cf. [25], Sect. 28 and Sect. 98).

In this paper we present a detailed exposition of this theory for the particular case of the cylindrical deformation of shells. The cylindrical deformation of shells is understood here in the sense of [15-17]. It is readily shown that the corresponding equations enjoy the same mathematical structure as that for plane deformation of rods. Accordingly, we shall speak of the theory of planar rods.

We begin in Chapter 2 by summerizing the basic equations for the plane strain problem of finite elasticity. In consistence with our interepretation of the planar rods, the basic relations summerized in Chapter 2 constitute the convenient starting point for the construction of the rod theory. Following [18] we next assume that: I) material fibres initially normal to the reference curve of the rod remain straight during the deformation, II) deformation of the rod is isochoric. We then show (Chapter 3) that the two-dimensional plane deformation of the rod consistent with the introduced assumptions is completely specified by the displacement field of the reference curve and the finite rotation vector characterizing the rotations of the initially normal fibres (rod cross-sections). Accordingly, the resulting one-dimensional kinematical model for the rod is identical to that in Reissner's theory [22] (cf. also [4-7,15-17]). However, the associated dynamical structure of the rod theory implied by the assumptions I) and II) is richer than previously has been examined in the relevant literature. Namely, as we show in Chapter 4 the one-dimensional strain energy function for the rod made of an incompressible hyperelastic material depends not only on the suitable strain measures but also on the derivative of strains.

The complete set of equations for the rod that are consistent with the underlying assumptions are derived in Chapter 5. Reduction of the general theory to special cases including the theory in which the shear deformation is constrained to vanish as well as the theory with the conventional dynamical structure is carried out in Chapter 6 and 7. We also exploit there the simplifying assumptions that are often not mentioned in the development of those types of the rod theories.

The explicit form of the constitutive relations for the general theory developed in this work are derived in Chapter 8. Finally, the buckling problem of the straight rods under axial compression is examined in Chapter 9. The comparison of the solutions to the problem obtained within general theory and its special cases provides insight into the relevance of various simplifications usually adopted in the literature.

The constitutive equations of the generality derived here, together with field equations governing finite strain deformation of rods provide the new theory which has been heretofore unavailable. In this respect our work generalizes many previous contributions in the literature on finite deformation of rods. In particular, in our development no assumptions concerning the magnitude of strains, rotations and even thickness of the rod are invoked except those that are implicit in the assumption I). Our objective in this paper was partly motivated by a desire to lay a fundation for eventual solutions to the problems like buckling of thick rods, short wave deformation of rods or necking and drawing in polymeric fibres under tension (cf. [11]).

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2. General cylindrical deformation of shells

Let $\{X_k\} = \{X_\beta, z\}$ be a fixed, right-handed Cartesian coordinate system in the space and let $\{\mathbf{i}_k\} = \{\mathbf{i}_\beta, \mathbf{k}\}$ denote the associated set of orthonormal base vectors. We adopt the convention that lower-case Latin indices have range 1,2,3, and that lower-case Greek indices have range 1,2, i.e. $i,j,\ldots = 1,2,3$ and $\alpha,\beta,\ldots = 1,2$. Such diagonally repeated indices are summed over their range. Vectors and tensors in the Euclidean space are denoted by bold-face letters. The inner product and the cross product of two vectors **a** and **b** are denoted by $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, respectively. The value of the (second order) tensor **T** at the vector **u** is denoted by **Tu**. The determinant $\det\mathbf{T}$ and the trace $\operatorname{tr}\mathbf{T}$ of **T** are defined asusual.

In this chapter we summarize the basic relations describing a general cylindrical deformation of a shell, i.e. a deformation of a cylindrical shell into another cylindrical shell such that all equations are independent of the coordinate along the generators of the shell. This is just a plane strain problem. We assume that the generators of the cylindrical shell are parallel to the coordinate z and thus (X_1, X_2) is the plane of the deformation. Accordingly, the undeformed configuration of the shell may be defined by

$$\mathbf{Y}(\xi^i) = \mathbf{X}(\xi^\alpha) + z\mathbf{k} , \tag{2.1}$$

where $\{\xi^i\} = \{\xi^\alpha, z\}$, $z \in [-a, +a]$, are material (convected) coordinates in the shell space and **X** is the position vector in the plane of the deformation, $\mathbf{X}\cdot\mathbf{k} = 0$. If $a = \infty$ we shall speak of the infinite cylindrical shell. The natural base vectors and the components of the metric tensor for the coordinate system $\{\xi^i\}$ are defined as usual

$$\begin{aligned} &\mathbf{g}_i = \mathbf{Y}_{,i} , \quad \mathbf{g}^i\cdot\mathbf{g}_j = \delta^i_j , \\ &g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j , \quad g^{ij} = \mathbf{g}^i\cdot\mathbf{g}^j , \quad g^{ik}g_{kj} = \delta^i_j . \end{aligned} \tag{2.2}$$

Here δ^i_j is the Kronecker symbol and a comma followed by the index i indicates partial differentiation with respect to ξ^i. In view of (2.1) we have

$$\mathbf{g}_\alpha = \mathbf{X}_{,\alpha} , \quad \mathbf{g}^\alpha\cdot\mathbf{g}_\beta = \delta^\alpha_\beta , \quad \mathbf{g}_3 = \mathbf{g}^3 = \mathbf{k} ,$$

$$g_{ij} = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & 1 \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} g^{\alpha\beta} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.3}$$

$$g^{\alpha\lambda} g_{\lambda\beta} = \delta^{\alpha}_{\beta} ,$$

$$g = \det g_{ij} = \det g_{\alpha\beta} > 0.$$

Clearly, all quantities defined in (2.3) are functions of the coordinates ξ^{α} only.

We next assume that external loadings, boundary conditions, and material properties are independent of the coordinate z. If, in addition, a is sufficiently large then the resulting deformation of the cylindrical shell will consist of a uniform stretch in the directions of the generators at the most. This is certainly the case for the infinite cylindrical shell (panel). Accordingly, the position vector of the particle whose the initial place was **Y** is of the form

$$\mathbf{y}(\xi^{i}) = \mathbf{x}(\xi^{\alpha}) + \lambda_z z\mathbf{k} , \tag{2.4}$$

where λ_z = const. is the out-of-plane stretch and **X** is the position vector of the particle with initial position **X**, $\mathbf{X}\cdot\mathbf{k} = 0$. Following Libai and Simmonds [15] we call (2.4) the cylindrical deformation of the shell. The natural base vectors and the components of the metric tensor associated with (2.4) are

$$\bar{g}_{\alpha} = \mathbf{x},_{\alpha} \;. \qquad \bar{g}^{\alpha}\cdot\bar{g}_{\beta} = \delta^{\alpha}_{\beta} ,$$

$$\bar{g}_3 = \lambda_z \mathbf{k} , \qquad \bar{g}^3 = \lambda_z^{-1}\mathbf{k} ,$$

$$\bar{g}_{ij} = \begin{bmatrix} \bar{g}_{\alpha\beta} & 0 \\ 0 & \lambda_z^2 \end{bmatrix} , \qquad \bar{g}^{ij} = \begin{bmatrix} \bar{g}^{\alpha\beta} & 0 \\ 0 & \lambda_2^{-2} \end{bmatrix} \tag{2.5}$$

$$\bar{g} = \det \bar{g}_{ij} = \lambda_z^2 \det \bar{g}_{\alpha\beta} \; .$$

Then the deformation gradient takes the form

$$F(\xi^\alpha) = \bar{g}_i \otimes g^i = \bar{g}_\alpha \otimes g^\alpha + \lambda_z k \otimes k \, , \tag{2.6}$$

and

$$J(\xi^\alpha) = \det F = \lambda_z \sqrt{\frac{\det \bar{g}_{\alpha\beta}}{g}} \, . \tag{2.7}$$

In (2.6) $a \otimes b$ denotes the tensor product of two vectors **a** and **b**. The transposition of **F** is given by $F^T = g^i \otimes \bar{g}_i$ and hence the right Cauchy-Green deformation tensor **C** takes the form

$$C(\xi^\alpha) = F^T F = \bar{g}_{ij}\, g^i \otimes g^j = \bar{g}_{\alpha\beta}\, g^\alpha \otimes g^\beta + \lambda_z^2\, k \otimes k \, . \tag{2.8}$$

Consequently, the principal invariants of **C** are

$$\begin{aligned} I_1 &= \mathrm{tr}C = g^{\alpha\beta}\, \bar{g}_{\alpha\beta} + \lambda_z^2 \, . \\ I_2 &= \frac{1}{2}\,[(\mathrm{tr}C)^2 - \mathrm{tr}C^2] = (g_{\alpha\beta}\, \bar{g}^{\alpha\beta} + \lambda_z^{-2})\, I_3 \, . \\ I_3 &= \det C = \frac{\bar{g}}{g} = \frac{\lambda_z^2 \det \bar{g}_{\alpha\beta}}{g} \, . \end{aligned} \tag{2.9}$$

The invariants I_k are not, however, independent for

$$g_{\alpha\beta}\, \bar{g}^{\alpha\beta}\, I_3 = \lambda_z^2\, g^{\alpha\beta}\, \bar{g}_{\alpha\beta} \, . \tag{2.10}$$

This can be shown by elementary tensor calculation [13]. We also observe that in the equations governing the cylindrical deformation of the shells the out-of-plane stretch λ_z plays solely a role of a parameter. Thus without loss in generality we may assume that $\lambda_z = 1$. This is indeed the case for the infinite cylindrical shell.

The main objective of this paper is to develop a theory of finite strain cylindrical deformatiton of shells made of a hyperelastic incompressible material. Accordingly, we shall assume that the deformation takes place without change in volume (isochoric deformation), i.e. $J(\xi^\alpha) = 1$. In view of (2.7) this assumption implies that $\det \bar{g}_{\alpha\beta} = g$. Moreover, from (2.9) and (2.10) we find out that

$$I_1(\xi^\alpha) = I_2(\xi^\alpha) = 1 + I, \tag{2.11}$$

$$I(\xi^\alpha) = g^{\alpha\beta}\,\bar{g}_{\alpha\beta} \geq 2 \;. \tag{2.12}$$

and, clearly, $I_3 = 1$. It is to be noted that in (2.11) the assumption $\lambda_z = 1$ has been incorporated. The incompressibility assumption is best satisfied by rubber-like materials which in turn are commonly assumed to be isotropic [2]. The mechanical response of such materials (hyperelastic, isotropic, incompressible) is governed by a strain energy density (per unit volume)

$$W = W(I_1, I_2) = C\hat{W}(I) \;, \tag{2.13}$$

where C is a positive constant of Young's modulus type. For late use we list below some proposals of W that accurately represent the response of rubber-like materials throughout the entire range of deformations [2,12]:

Mooney-Rivlin material (C_1, C_2 - material constants)

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) = C(I - 2),\; C = C_1 + C_2 \tag{2.14}$$

Hart-Smith material (C, k_1, k_2 - material constants)

$$W = C\,\{\int e^{k_1(I_1-3)^2} dI_1 + k_2 \ln(\frac{I_2}{3})\} =$$

$$= C\,\{\int e^{k_1(I-2)^2} dI + k_2 \ln(\frac{1+I}{3})\} \;, \tag{2.15}$$

Biomaterial (α, β - material constants)

$$W = \frac{\beta}{2\alpha}\,[e^{\alpha(I_2-3)} - 1] = C\,\{\frac{1}{\alpha}\,[e^{\alpha(I-2)} - 1]\},\; C = \frac{1}{2}\beta \tag{2.16}$$

3. Nonlinear rod model

In this chapter we clearly state the assumptions that play a central role to the subsequent analysis. These assumptions enable to reduce the two-dimensional problem of the plane deformation of rods to the one-dimensional theory with an underlaying kinematical structure identical to that in the theories developed by different means by Antman [4,5], Reissner [22], Libai and Simmonds [16,17] and others. The basic notation use here are shown in Fig. 1.

It is convenient and desirable for analitical simplicity to take the material coordinate system $\{\xi^\alpha\} = \{s, \xi\}$ to be normal one in the undeformed configuration of the rod. Then the position vector $\mathbf{X}$ takes the form

$$\mathbf{X}(s,\xi) = \mathbf{r}(s) + \xi \mathbf{e}_2(s), \qquad \xi \in [-h_o^-, +h_o^+] \ . \tag{3.1}$$

Here

$$\mathbf{r}(s) = \mathbf{X}(s,0) = X_1(s)\mathbf{i}_1 + X_2(s)\mathbf{i}_2 \ . \tag{3.2}$$

is the position vector of a reference curve c of the rod and s, $s \in [0,l]$, denotes the arc length parameter along c. The initial rod thickness (possibly variable) is defined by $h_o = h_o^+ + h_o^-$. The specific choice of the reference curve c may be dictated by a problem at the hands. In particular, for relatively thick rod or in the analysis of contact problems either the upper $\mathbf{X}^+(s) = \mathbf{X}(s, h_o^+)$ or lower $\mathbf{X}^-(s) = \mathbf{X}(s, -h_o^-)$ rod face may be choosen as the reference curve. In general considerations, however, it is preferable to leave it unspecified. The unit vectors tangent and normal to c are given by

$$\begin{aligned} \mathbf{e}_1(s) &= \mathbf{r}' = \cos\varphi \ \mathbf{i}_1 + \sin\varphi \ \mathbf{i}_2 \ , \\ \mathbf{e}_2(s) &= \mathbf{k} \times \mathbf{e}_1 = -\sin\varphi \ \mathbf{i}_1 + \cos\varphi \ \mathbf{i}_2 \ . \end{aligned} \tag{3.3}$$

where a prime indicates the differentiation with respect to the are length parameter and the angle $\varphi(s)$ is defined in Fig. 1. The curvature of c is

$$K(s) = \mathbf{r}''\cdot\mathbf{e}_2 = \varphi' \ , \tag{3.4}$$

so that

$$\mathbf{e}_1' = K\mathbf{e}_2 \ , \qquad \mathbf{e}_2' = -K\mathbf{e}_1 \ . \tag{3.5}$$

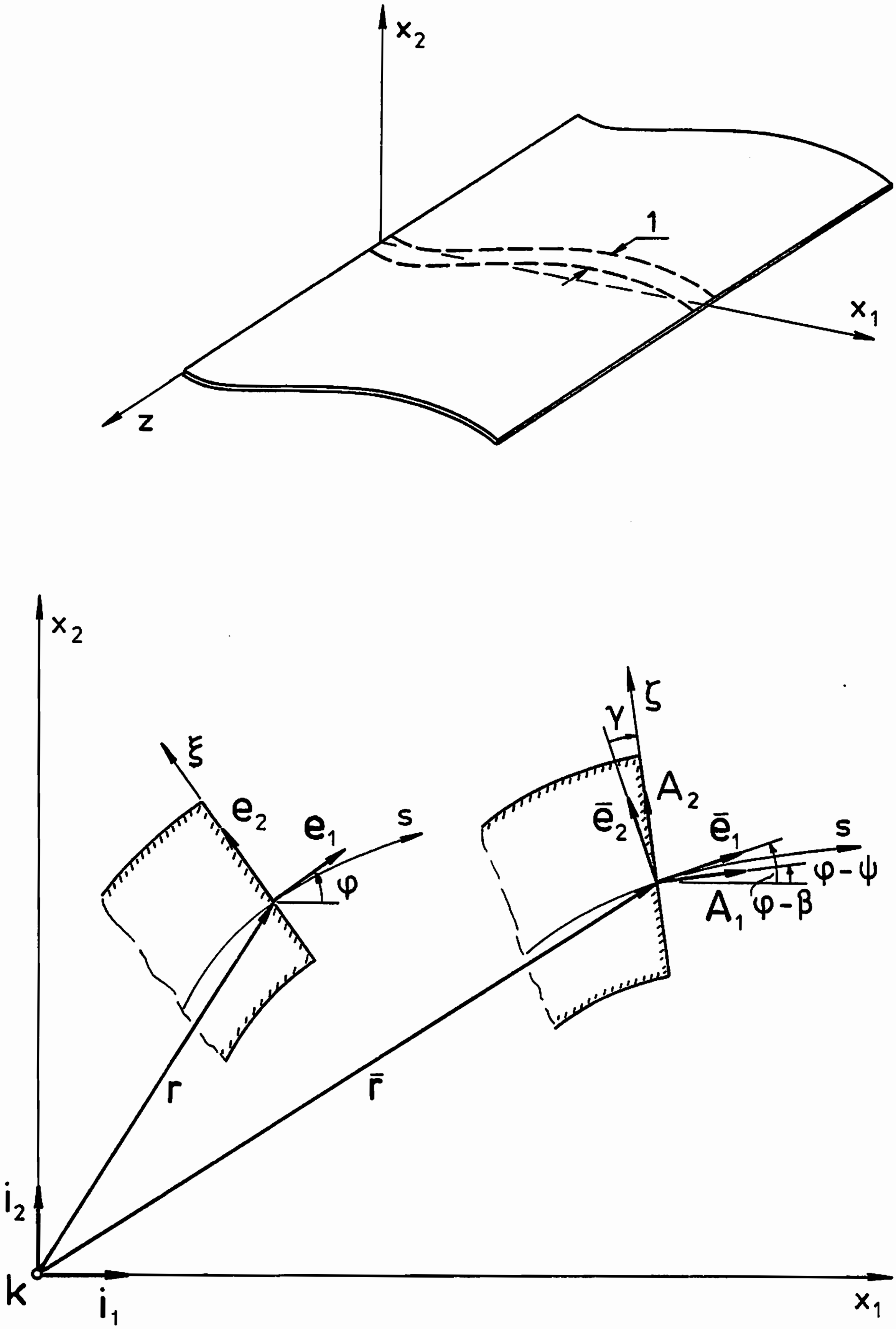

Fig. 1 Geometry of planar rod, a) planar rod as a section of unit width of a cylindrical shell, b) coordinate system and base vectors

From (3.1), (3.5), (2.2) and (2.3) we obtain the known formulae

$$g_1 = X' = \mu e_1 , \qquad g_2 = X_{,\xi} = e_2 ,$$

$$g^1 = \mu^{-1} e_1 , \qquad g^2 = e_2 , \tag{3.6}$$

$$g_{\alpha\beta} = \begin{bmatrix} \mu^2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g^{\alpha\beta} = \begin{bmatrix} \mu^{-2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad g = \mu^2$$

where $(\ )_{,\xi}$ stands for the partial derivative with respect to the normal coordinate ξ and

$$\mu(s,\xi) = 1 - K\xi . \tag{3.7}$$

The image $\bar{c}$ of c in the deformed configuration of the rod is a material curve specified by its position vector

$$\bar{r}(s) = x(s,0) = x_1(s)\, i_1 + x_2(s)\, i_2 . \tag{3.8}$$

The unit vectors tangent and normal to $\bar{c}$, respectively, and the curvature of are given by

$$\bar{e}_1(s) = \lambda^{-1}\, \bar{r}' = \cos\bar{\varphi}\, i_1 + \sin\bar{\varphi}\, i_2 ,$$

$$\bar{e}_2(s) = k \times \bar{e}_1 = - \sin\bar{\varphi}\, i_1 + \cos\bar{\varphi}\, i_2 . \tag{3.9}$$

$$\bar{K}(s) = \lambda^{-2}\, \bar{r}'' \cdot \bar{e}_2 = \lambda^{-1} \bar{\varphi}' , \tag{3.10}$$

so that

$$\bar{e}_1' = \lambda\bar{K}\, \bar{e}_2 , \qquad \bar{e}_2' = - \lambda\bar{K}\, \bar{e}_1 . \tag{3.11}$$

where

$$\lambda(s) = \sqrt{\bar{r}' \cdot \bar{r}'} , \tag{3.12}$$

denotes the stretch of the reference curve. By virtue of the obvious regularity conditions, $0 < \lambda < +\infty$.

With each point of the deformed reference curve $\bar{c}$ we next associate an orthonormal base $\{A_\beta\}$ that rigidly rotates during the rod deformation and coincides with $\{e_\beta\}$ in the undeformed configuration (see Fig. 1)

$$A_1(s) = Qe_1 = \cos\varphi\ e_1 - \sin\varphi\ e_2 ,$$
$$A_2(s) = Qe_2 = \sin\varphi\ e_1 + \cos\varphi\ e_2 , \tag{3.13}$$

where

$$Q(s) = A_1 \otimes e_1 + A_2 \otimes e_2 . \tag{3.14}$$

is the proper orthogonal tensor, i.e. the element of the Lie group SO(2). The geometric meaning of the basis $\{A_\beta\}$ will become clear later on. Introducing the shear angle $\gamma(s) = A_2 \cdot \bar{e}_2$, $|\gamma(s)| < \frac{\pi}{2}$. the vectors A_β can alternatively be expressed in the form

$$A_1 = \cos\gamma\ \bar{e}_1 - \sin\gamma\ \bar{e}_2 , \qquad A_2 = \sin\gamma\ \bar{e}_1 + \cos\gamma\ \bar{e}_2 . \tag{3.15}$$

The differentiation of (3.15) with the use of (3.11) gives

$$A_1' = \tilde{K}A_2 , \qquad A_2' = - \tilde{K}A_1 . \tag{3.16}$$

where

$$\tilde{K}(s) = \lambda\bar{K} - \gamma' = \theta' = \bar{\varphi}' - \gamma' . \tag{3.17}$$

In this work we intend to formulate a finite strain theory of rods which consists of the displacement field $\mathbf{u}$ of the reference curve and the finite rotation vector $\boldsymbol{\phi}$ corresponding to Q as the only independent kinematical variables:

$$\mathbf{u}(s) = \bar{\mathbf{r}} - \mathbf{r} = u(s)\mathbf{i}_1 + w(s)\mathbf{i}_2 , \qquad \boldsymbol{\phi}(s) = - \phi(s)\mathbf{k} . \tag{3.18}$$

The suitable strain measures for this rod model have been introduced in [5,16,22]. Our derivation given below slightly differes from that in those papers. We first note that since Q is the orthogonal tensor, $Q'Q^T$ must be skew-symmetric one. Indeed, from (3.16), (3.5) and (3.14) one gets

$$Q'Q^T = -(\tilde{K} - K)\, A_1 \wedge A_2 . \tag{3.19}$$

Here $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}$ denotes the exterior product of two vectors $\mathbf{a}$ and $\mathbf{b}$. We also recall that the axial vector, $\mathbf{w}$ say, of the skew-symmetric tensor $\mathbf{a} \wedge \mathbf{b}$ is defined by

$$(\mathbf{a} \wedge \mathbf{b})\mathbf{v} = \mathbf{w} \times \mathbf{v} \quad \text{for any vector } \mathbf{v}. \tag{3.20}$$

It can be shown that $\mathbf{w} = -\mathbf{a} \times \mathbf{b}$. The strain measures are now defined from $(3.9)_1$ and (3.19). They consist of the stretching vector

$$\boldsymbol{\varepsilon}(s) = \bar{\mathbf{r}}' - \mathbf{A}_1 = \varepsilon \mathbf{A}_1 + \eta \mathbf{A}_2 , \tag{3.21}$$

and the bending vector

$$\boldsymbol{\varkappa}(s) = -\varkappa(s)\mathbf{k} , \tag{3.22}$$

defined as the axial vector of the skew-symmetric tensor $Q'Q^T$. From $(3.9)_1$, (3.15) and (3.19) it follows that

$$\begin{aligned} \varepsilon(s) &= \lambda\cos\gamma - 1, \qquad \eta(s) = \lambda\sin\gamma , \\ \varkappa(s) &= -(\tilde{K} - K) = -(\lambda\bar{K} - \gamma' - K) . \end{aligned} \tag{3.23}$$

According to Fig. 1 we have $\bar{\varphi} = \varphi - \psi + \gamma$ and hence using (3.18), (3.13) and (3.17) we obtain the associated strain-displacement relations

$$\begin{aligned} \varepsilon(s) &= \bar{\mathbf{r}}'\cdot A_1 - 1 = x_1' \cos(\varphi - \psi) + x_2' \sin(\varphi - \psi) - 1 = \\ &= u'\cos(\varphi - \psi) + w'\sin(\varphi - \psi) + \cos\psi - 1, \\ \eta(s) &= \bar{\mathbf{r}}'\cdot A_2 = -x_1'\sin(\varphi - \psi) + x_2'\cos(\varphi - \psi) = \\ &= -u'\sin(\varphi - \psi) + w'\cos(\varphi - \psi) + \sin\psi , \\ \varkappa(s) &= \psi' . \end{aligned} \tag{3.24}$$

In passing it is worthwhile to note that alternative sets of strain measures may also be defined (cf. [15,16]).

We emphasize that the rod model with **u** and ϕ as the only independent kinematical variables may be given different two-dimensional interpretation. In other words, virtually different assumptions may lead to the same one-dimensional kinematical model for the rod. The underlying assumptions in our analysis are:

I) material fibres initially normal to the reference curve remain straight (but not necessarily normal) during the rod deformation,

II) deformation of the rod is isochoric, $J(s,\xi) = 1$.

Of these two only the first assumption is of approximate nature while the second one reflects merely the real property of many materials capable to undergo finite deformation. We shall not attempt to legitimate reliability of the above assumptions (see discussion given in chapter 9). Our aim is rather to investigate in full details their implications.

According to the assumption I) the current position vector of the particle whose initial place was **X** may be expressed in the form

$$\mathbf{x}(s,\xi) = \bar{\mathbf{r}}(s) + \zeta(s,\xi)\, \mathbf{A}_2(s) \ , \tag{3.25}$$

where a scalar-valued function $\zeta(s,\xi)$ must satisfy the condition $\zeta(s,0) = 0$. The geometric meaning of (3.25) is obvious. The unit vector $\mathbf{A}_2(s)$ or, equivalently, the angle $\phi(s)$ characterize the rotation of the initially normal fibres (rod cross-sections), while $\zeta(s,\xi)$ accommodates an arbitrary transverse normal deformation of the rod. According to (3.21)

$$\bar{\mathbf{r}}' = \nu\mathbf{A}_1 + \eta\mathbf{A}_2 \ . \qquad \nu(s) = 1 + \varepsilon = \lambda\cos\gamma \ , \tag{3.26}$$

and hence the differentiation of (3.25) with the use of (3.26) and (3.16) yields

$$\begin{aligned} \bar{\mathbf{g}}_1 &= \mathbf{x}' = (\nu - \bar{K}\zeta)\mathbf{A}_1 + (\eta + \zeta')\mathbf{A}_2 \ . \\ \bar{\mathbf{g}}_2 &= \mathbf{x}_{,\xi} = \zeta_{,\xi}\, \mathbf{A}_2 \ . \end{aligned} \tag{3.27}$$

Solving next the system of equations $\bar{\mathbf{g}}^\alpha \cdot \bar{\mathbf{g}}_\beta = \delta^\alpha_\beta$ we obtain

$$\bar{g}^1 = (\nu - \tilde{K}\zeta)^{-1} A_1 .$$
$$\bar{g}^2 = - (\zeta,_\xi)^{-1} [(\nu - \tilde{K}\zeta)^{-1} (\eta + \zeta')A_1 - A_2] . \quad (3.28)$$

From (3.27) and (3.28) the components of the metric tensor follow

$$\bar{g}_{11} = (\nu - \tilde{K}\zeta)^2 + (\eta + \zeta')^2 ,$$
$$\bar{g}_{22} = (\zeta,_\xi)^2 . \quad (3.29)$$
$$\bar{g}_{12} = \bar{g}_{21} = (\eta + \zeta')\zeta,_\xi .$$
$$\bar{g}^{11} = (\nu - \tilde{K}\zeta)^{-2} ,$$
$$\bar{g}^{22} = (\zeta,_\xi)^2 [(\nu - \tilde{K}\zeta)^{-2} (\eta + \zeta')^2 + 1] , \quad (3.30)$$
$$\bar{g}^{12} = \bar{g}^{21} = - (\zeta,_\xi)^{-1} (\nu - \tilde{K}\zeta)^{-2} (\eta + \zeta') ,$$

and hence

$$\bar{g} = \det \bar{g}_{\alpha\beta} = (\nu - \tilde{K}\zeta)^2 (\zeta,_\xi)^2 . \quad (3.31)$$

In order to complete the kinematical considerations it remains to specify the function $\zeta(s,\xi)$. To this end we firt note that in view of (3.31), (3.6) and (2.7) $J(s,\xi) = (\nu - \tilde{K}\zeta)\zeta,_\xi(1 - K\xi)^{-1}$ and hence the assumption II) implies the following first order differential equations for ζ

$$(\nu - \tilde{K}\zeta) \frac{\partial\zeta}{\partial\xi} = 1 - K\xi. \quad (3.32)$$

The integration of (3.32) with the use of the initial condition $\zeta(s,0) = 0$ yields

$$\tilde{K}\zeta^2 - 2\nu\zeta + 2\xi - K\xi^2 = 0 . \quad (3.33)$$

If $\tilde{K} = 0$ it immediately follows that

$$\zeta(s,\xi) = \nu^{-1}(\xi - \tfrac{1}{2} K\xi^2). \quad (3.34)$$

However, this is a very special case and we shall assume henceforth that $\tilde{K} \neq 0$. Then the solution of the quadratic algebraic equation (3.34) satisfying the condition $\zeta(s,0) = 0$ is

$$\zeta(s,\xi) = \nu\tilde{K}^{-1} \{1 - \sqrt{1 - \nu^2\tilde{K}(2\xi - K\xi^2)}\} \ , \tag{3.35}$$

or, written in more compact form

$$\zeta(s,\xi) = \tilde{K}^{-1}(\nu - \sqrt{\Lambda}) \ , \tag{3.36}$$

where

$$\Lambda(s,\xi) = \nu^2 + \tilde{K}K^{-1}(\mu^2 - 1) \qquad \text{if } K \neq 0 \ , \tag{3.37}$$

$$\Lambda(s,\xi) = \nu^2 - 2\tilde{K}\xi = \nu^2 + 2\varkappa\xi \qquad \text{if } K = 0 \ . \tag{3.38}$$

Introducing now (3.36) into (3.29) and (3.30) and making use of (3.32) we finally obtain

$$\bar{g}_{\alpha\beta} = \begin{pmatrix} 1 + (\eta + \zeta')^2 & \dfrac{\mu(\eta + \zeta')}{\sqrt{\Lambda}} \\ \dfrac{\mu(\eta + \zeta')}{\sqrt{\Lambda}} & \dfrac{\mu^2}{\Lambda} \end{pmatrix} \tag{3.39}$$

$$\bar{g}^{\alpha\beta} = \begin{pmatrix} \dfrac{1}{\Lambda} & -\dfrac{\eta + \zeta'}{\mu\sqrt{\Lambda}} \\ -\dfrac{\eta + \zeta'}{\mu\sqrt{\Lambda}} & \dfrac{1}{\mu^2}[\Lambda + (\eta + \zeta')^2] \end{pmatrix} \tag{3.40}$$

with ζ' to be evaluated from (3.36).

In this way we have shown that the two-dimensional (plane) deformation of the rod is completely specified by the displacement field $\mathbf{u}(s)$ of the rod reference curve and the rotation vector $\boldsymbol{\phi}(s)$ specifying the rotations of the rod cross-sections. The only assumptions made here are those I) and II). This result represents a substantial generalization of many earlier investigations on the subject [9,10,15]. Moreover, our detailed treatment of the problem reveals that the one-dimensional kinematical structure for the rod implied by the assumptions I) and II) is identical to that in Reissner's theory [22]. However, the resulting dynamical structure is richer than in the rod theories previously investigated [5-7,16,17,22,24]. In order to see this we now examine the form of the one-dimensional strain energy function for the rod.

4. Strain energy function

The one-dimenional strain energy function (per unit length of the undeformed reference curve) is simply obtained by integration the strain energy density (2.13) through the rod thickness

$$\Phi = Ch_o \int_{-}^{+} \hat{W}(I)\, \mu\, d\hat{\xi} \ , \qquad \int_{-}^{+} \equiv \int_{-\alpha^-}^{+\alpha^+} \tag{4.1}$$

where we have introduced the dimensionless normal coordinate

$$\hat{\xi} = \frac{\xi}{h_o} \in [-\alpha^-, +\alpha^+] \ , \qquad \alpha^{\pm} = \frac{h_o^{\pm}}{h_o} \leq 1 \ . \tag{4.2}$$

We also define the dimensionless thickness parameter

$$\epsilon_h(s) = - h_o K = \frac{h_o}{R}, \qquad R(s) = - K^{-1} \ , \tag{4.3}$$

with R being the radious of curvature of the undeformed reference curve. Clearly, if $K = 0$ then $\epsilon_h = 0$. From (3.39), (3.6), (2.12) and (3.36) we find out that

$$I(s,\hat{\xi}) = \mu^{-2}\Lambda + \mu^2\Lambda^{-1} + \mu^{-2}(\eta + \zeta')^2 \ , \tag{4.4}$$

where

$$\Lambda(s,\hat{\xi}) = (1 + \varepsilon)^2 + (1 + {}_{h}^{-1}h_o\varkappa)\ (\mu^2 - 1) \ , \tag{4.5}$$

$$\mu(s,\hat{\xi}) = 1 + \epsilon_h\hat{\xi} \ , \tag{4.6}$$

$$\zeta(s,\hat{\xi}) = h_o(\epsilon_h + h_o\varkappa)^{-1}(\sqrt{\Lambda} - \varepsilon - 1) \ , \tag{4.7}$$

if $K \neq 0$, and

$$\Lambda(s,\xi) = (1 + \varepsilon)^2 + 2h_o\varkappa\hat{\xi} \ , \qquad \mu = 1, \tag{4.8}$$

$$\zeta(s,\hat{\xi}) = h_o(h_o\varkappa)^{-1}\ (\sqrt{\Lambda} - \varepsilon - 1) \ , \tag{4.9}$$

if $K = 0$. The basic conclusion that we can read off from (4.4) is that the strain energy (4.1) is the function not only of the strains ε, η and $\varkappa$ but also of the gradient of ε and $\varkappa$, that is

$$\Phi = \Phi(\varepsilon, \eta, \varkappa, \varepsilon', \varkappa'). \tag{4.10}$$

The detailed examination of the strain energy function (4.10) and the associated constitutive equations for the rod we postpone until the Chapter 8. Here we only note that the rod whose mechanical response is governed by the strain energy function of the form (4.10) may be viewed as the one-dimensional counterpart of the non-simple material in the finite elasticity (cf. [25], Sects. 28,98).

5. Governing equations

The field equations and boundary conditions that are consistent with the introduced assumptions may now be obtained from the two-dimensional principle of virtual work. Let $f(s,\xi)$, $p^{\pm}(s)$ and $t^{*}(\xi)$ denote the body force, the external forces acting on the rod faces and the external force applied to the rod ends, respectively, all measured with respect to the undeformed configuration. For the hyperelastic rod the two-dimensional principle of virtual work takes the form

$$\delta \iint_{0-}^{1+} W\mu \, d\xi = \int_0^1 \left(\int_-^+ f\cdot\delta x\mu \, d\xi + \mu^+ p^+\cdot\delta x^+ + \mu^- p^-\cdot\delta x^- \right) ds + \tag{5.1}$$
$$+ \left[\int_-^+ t^{*}\cdot\delta x d\xi \right]_0^1 .$$

where $x^{\pm}(s) = x(s, \pm h_0^{\pm})$ and $\mu^{\pm}(s) = \mu(s, \pm h_0^{\pm})$. From (3.13) and (3.18) we have

$$\delta A_\beta = \delta\phi \times A_\beta , \tag{5.2}$$

and hence the virtual change of the position vector (3.25) is

$$\delta x = \delta\bar{r} + \zeta\delta\phi \times A_2 + (\zeta_\varepsilon \delta\varepsilon + \zeta_\varkappa \delta\varkappa) \, A_2 \tag{5.3}$$

Here and henceforth a subscript indicates the differentiation with respect to the underlying strains. Introducing (5.3) into (5.1) and making use of (4.1) the reduced one-dimensional principle of virtual work takes the form

$$\int_0^1 \delta\Phi ds = \int_0^1 (p\cdot\delta\bar{r} + l\cdot\delta\phi + f\delta\varepsilon + k\delta\varkappa) ds + \tag{5.4}$$
$$+ \left[N^{*}\cdot\delta\bar{r} + M^{*}\cdot\delta\phi + n^{*}\delta\varepsilon + m^{*}\delta\varkappa \right]_0^1 .$$

where

$$\mathbf{p}(s) = \int_{-}^{+} \mathbf{f}\mu d\xi + \mu^{+}\mathbf{p}^{+} + \mu^{-}\mathbf{p}^{-} \equiv p_1\mathbf{A}_1 + p_2\mathbf{A}_2 \; . \qquad (5.5)$$

$$\mathbf{l}(s) = \mathbf{A}_2 \times \left(\int_{-}^{+} \zeta\mathbf{f}\mu d\xi + \mu^{+}\zeta^{+}\mathbf{p}^{+} + \mu^{-}\zeta^{-}\mathbf{p}^{-}\right) \equiv - l\mathbf{k}, \qquad (5.6)$$

$$f(s) = \mathbf{A}_2\cdot\left(\int_{-}^{+} \zeta_{\varepsilon}\mathbf{f}\mu d\xi + \mu^{+}\zeta_{\varepsilon}^{+}\mathbf{p}^{+} + \mu^{-}\zeta_{\varepsilon}^{-}\mathbf{p}^{-}\right), \qquad (5.7)$$

$$k(s) = \mathbf{A}_2\cdot\left(\int_{-}^{+} \zeta_{\varkappa}\mathbf{f}\mu d\xi + \mu^{+}\zeta_{\varkappa}^{+}\mathbf{p}^{+} + \mu^{-}\zeta_{\varkappa}^{-}\mathbf{p}^{-}\right), \qquad (5.8)$$

$$\mathbf{N}^{*} = \int_{-}^{+} \mathbf{t}^{*} d\xi \equiv N^{*}\mathbf{A}_1 + Q^{*}\mathbf{A}_2 \; . \qquad (5.9)$$

$$\mathbf{M}^{*} = \mathbf{A}_2 \times \int_{-}^{+} \zeta\mathbf{t}^{*} d\xi \equiv - M^{*}\mathbf{k}, \qquad (5.10)$$

$$n^{*} = \mathbf{A}_2\cdot\left(\int_{-}^{+} \zeta_{\varepsilon}\mathbf{t}^{*} d\xi\right), \qquad (5.11)$$

$$m^{*} = \mathbf{A}_2\cdot\left(\int_{-}^{+} \zeta_{\varkappa}\mathbf{t}^{*} d\xi\right), \qquad (5.12)$$

are the resulting external distributed and end loadings. Furthermore, according to (4.10)

$$\delta\Phi = N\delta\varepsilon + Q\delta\eta + M\delta\varkappa + n\delta\varepsilon' + m\delta\varkappa' \; , \qquad (5.13)$$

where the resulting internal forces and couples are defined by

$$N = \Phi_{\varepsilon}(\mathfrak{e}), \qquad Q = \Phi_{\eta}(\mathfrak{e}), \qquad M = \Phi_{\varkappa}(\mathfrak{e}) \qquad (5.14)$$

$$n = \Phi_{\varepsilon'}(\mathfrak{e}), \qquad m = \Phi_{\varkappa'}(\mathfrak{e}), \qquad (5.15)$$

with $\mathfrak{e} = (\varepsilon, \eta, \varkappa, \varepsilon', \varkappa')$. Introducing (5.13) into (5.4) and applying subsequently integration by parts we obtain

$$\int_0^1 \{(N - n' - f)\delta\varepsilon + Q\delta\eta + (M - m' - k)\delta\varkappa\}ds -$$
$$- \int_0^1 (\mathbf{p}\cdot\delta\bar{\mathbf{r}} + \mathbf{l}\cdot\delta\boldsymbol{\phi})ds - \tag{5.16}$$
$$+ [- \mathbf{N}^*\cdot\delta\bar{\mathbf{r}} - \mathbf{M}^*\cdot\delta\boldsymbol{\phi} + (n - n^*)\delta\varepsilon + (m - m^*)\delta\varkappa]_0^1 = 0.$$

Next, from (3.21), (3.22) and (5.2) we have

$$\delta\varepsilon\mathbf{A}_1 + \delta\eta\mathbf{A}_2 = \delta\boldsymbol{\varepsilon} - \delta\boldsymbol{\phi} \times \boldsymbol{\varepsilon} = \delta\bar{\mathbf{r}}' - \delta\boldsymbol{\phi} \times \bar{\mathbf{r}}' ,$$
$$- \delta\varkappa\mathbf{k} = \delta\boldsymbol{\varkappa} = \delta\boldsymbol{\phi}' . \tag{5.17}$$

With the use of (5.17) the principle of virtual work (5.16) may finally be expressed in the form

$$- \int_0^1 \{(\mathbf{N}' + \mathbf{p})\cdot\delta\bar{\mathbf{r}} + (\mathbf{M}' + \bar{\mathbf{r}}' \times \mathbf{N} + \mathbf{l})\cdot\delta\boldsymbol{\phi}\}ds +$$
$$+ [(\mathbf{N} - \mathbf{N}^*)\cdot\delta\bar{\mathbf{r}} + (\mathbf{M} - \mathbf{M}^*)\cdot\delta\boldsymbol{\phi} + \tag{5.18}$$
$$+ (n - n^*)\delta\varepsilon + (m - m^*)\delta\varkappa]_0^1 = 0 ,$$

where the resultant force and couple vectors are defined by

$$\mathbf{N}(s) = (N - n' - f)\mathbf{A}_1 + Q\mathbf{A}_2 .$$
$$\mathbf{M}(s) = - (M - m' - k)\mathbf{k} . \tag{5.19}$$

From (5.18) we read of the local equilibrium equations

$$\mathbf{N}' + \mathbf{p} = 0 \,,$$
$$\mathbf{M}' + \bar{\mathbf{r}}' \times \mathbf{N} + \mathbf{l} = 0 \,, \tag{5.20}$$

and the associated boundary conditions at s = 0,1

$$\begin{aligned} \mathbf{N} &= \mathbf{N}^* \quad &\text{or} \quad \bar{\mathbf{r}} &= \bar{\mathbf{r}}^* \,, \\ \mathbf{M} &= \mathbf{M}^* \quad &\text{or} \quad \boldsymbol{\phi} &= \boldsymbol{\phi}^* \,, \\ n &= n^* \quad &\text{or} \quad \varepsilon &= \varepsilon^* \,, \\ m &= m^* \quad &\text{or} \quad \varkappa &= \varkappa^* \,. \end{aligned} \tag{5.21}$$

Here the asterisk stands for the quantity prescribed at the rod ends. We next note that on making use of (5.17) the boundary terms in (5.18) may alternatively be expressed in the form

$$[(\mathbf{N} - \mathbf{N}^*)\cdot\delta\bar{\mathbf{r}} + \{\mathbf{M} - \bar{\mathbf{r}}' \times (\mathbf{n} - \mathbf{n}^*) - \mathbf{M}^*\}\cdot\delta\boldsymbol{\phi} + \\ + (\mathbf{n} - \mathbf{n}^*)\cdot\delta\bar{\mathbf{r}}' + (\mathbf{m} - \mathbf{m}^*)\cdot\delta\boldsymbol{\phi}']_0^1 \,. \tag{5.22}$$

where

$$\mathbf{n} = n\mathbf{A}_1 \,, \qquad \mathbf{m} = -\, m\mathbf{k} \,. \tag{5.23}$$

From (5.22) different though equivalent form of the boundary conditions follows.

The equilibrium equations (5.20) and boundary conditions (5.21) written in the component form are

$$\begin{aligned} &(N - n' - f)' - (K - \varkappa)Q + p_1 = 0 \,, \\ &Q' + (K - \varkappa)(N - n' - f) + p_2 = 0 \,, \\ &(M - m' - k)' - (1 + \varepsilon)Q + \eta(N - n' - f) + l = 0 \,, \end{aligned} \tag{5.24}$$

and

$$N - n' - f = N^* \quad \text{or} \quad (x_1 - x_1^*)\cos\theta + (x_2 - x_2^*)\sin\theta = 0 \,,$$

$$Q = Q^* \quad \text{or} \quad -(x_1 - x_1^*)\sin\theta + (x_2 - x_2^*)\cos\theta = 0 \ ,$$

$$M - m' - k = M^* \quad \text{or} \quad \psi = \psi^* \ , \tag{5.25}$$

$$n = n^* \quad \text{or} \quad \varepsilon = \varepsilon^* \ ,$$

$$m = m^* \quad \text{or} \quad \varkappa = \varkappa^* \ ,$$

where

$$x_1 = X_1 + u, \qquad x_2 = X_2 + w, \qquad \theta = \varphi - \psi \ . \tag{5.26}$$

Using the strain-displacement relations (3.24)the geometric boundary conditions $(5.25)_{4,5}$ may equivalently be expressed in terms of the displacements

$$x_1' \cos\theta + x_2' \sin\theta - 1 = \varepsilon^* \ ,$$
$$\psi' = \varkappa^* \tag{5.27}$$

If the rod has a closed form (e.g. circular ring) the boundary conditions must be replaced by suitable periodicity conditions.

The governing equations for our rod model consist of the equilibrium equations (5.24), the constitutive equations (5.14) and (5.15), the kinematical relations (3.24) and the boundary conditions (5.25). We note that the kinematical relations (3.24) may be inverted to yield

$$x_1'(s) = (1 + \varepsilon) \cos(\varphi - \psi) - \eta \sin(\varphi - \psi) \ ,$$

$$x_2'(s) = (1 + \varepsilon) \sin(\varphi - \psi) + \eta \cos(\varphi - \psi) \ , \tag{5.28}$$

$$\psi'(s) = \varkappa \ .$$

The equilibrium equations (5.24) and the constitutive relations (5.14), (5.15) constitute the determinated system of equations for the strains ε, η and $\varkappa$. If these equations could be solved for the strains then the displacements may be found by integration of the relations (5.28).

6. Non-simple unshearable rod model

A particular version of the rod theory developed in the previous chapters is obtained if the shear is constrained to vanish, i.e. $\gamma = 0$. This is equivalent to replacing the assumption I) by more restrictive assumption:

Ia) material fibres initially normal to the reference curve remain normal to it after deformation.

Then

$$A_\beta = e_\beta , \qquad \psi = \beta = \varphi - \bar{\varphi}, \qquad \eta = 0 , \tag{6.1}$$

what implies the following form of the strain measures

$$\varepsilon(s) = \lambda - 1, \qquad \varkappa(s) = -(\lambda\bar{K} - K) = \beta' . \tag{6.2}$$

All kinematical relations for this particular case follow immediately from that derived in the chapter 3 and 4 by introducing the constraints (6.1). In particular, the expression (4.3) for I takes now the form

$$I(s,\hat{\xi}) = \mu^{-2}\Lambda + \mu^{2}\Lambda + \mu^{-2}(\zeta')^{2} , \tag{6.3}$$

with $\zeta(s,\hat{\xi})$ and $\Lambda(s,\hat{\xi})$ given by (4.7) and (4.5) if $K \neq 0$ and by (4.9) and (4.8) if $K = 0$. As a result the one-dimensional strain energy function is of the form

$$\Phi = \Phi(\varepsilon, \varkappa, \varepsilon', \varkappa') , \tag{6.4}$$

and, consequently, the constitutive equations are

$$N = \Phi_{\varepsilon}(\mathfrak{e}), \qquad M = \Phi_{\varkappa}(\mathfrak{e}) , \tag{6.5}$$

$$n = \Phi_{\varepsilon'}(\mathfrak{e}), \qquad m = \Phi_{\varkappa'}(\mathfrak{e}) , \tag{6.6}$$

with $\mathfrak{e} = (\varepsilon, \varkappa, \varepsilon', \varkappa')$. The transverse shear force Q is no longer determined by the constitutive relation but it plays the role of a Lagrange multiplier in the equations in which it appears. The vector form (5.20) of the equilibrium equations and of the boundary conditions (5.21) remain valid. However, the component form (5.24) of the equilibrium equations reduces now to the form

$$(N - n' - f)' - (K - \varkappa)Q + p_1 = 0 ,$$

$$Q' + (K - \varkappa)(N - n' - f) + p_2 = 0 , \qquad (6.7)$$

$$(M - m' - k)' - (1 + \varepsilon)Q + 1 = 0 .$$

These equations may be reduced to two in number or, what means precisely the same, the two vector equilibrium equations (5.20) may be reduced to one vector equation. This is obvious, since the finite rotation vector $\boldsymbol{\phi} = \boldsymbol{\beta} = - \beta\mathbf{k}$ is no longer the independent kinematical variable. It may be expressed as a function of the position vector $\bar{\mathbf{r}}$. Indeed, from (3.11) we have

$$\cos\beta = \lambda^{-1}\bar{\mathbf{r}}'\cdot\mathbf{e}_1 . \qquad \lambda = \sqrt{\bar{\mathbf{r}}'\cdot\bar{\mathbf{r}}'} . \qquad (6.8)$$

Thus for the unshearable rod model the position vector $\bar{\mathbf{r}}$ (or the displacement field $\mathbf{u}$) is the only independent kinematical variable.

7. Simple rod models

The another particular versions of the general theory with either shear deformation accounted for or not may be obtained if in addition to the previous assumptions we suppose that:

III) deformation of the rod is sufficiently smooth in the sense that the contribution to the strain energy function Φ of all terms containing ζ' is negligible.

For the isotropic material being of primary interest to us this simply means that the expression (4.3) for I may be assumed in the simplified form

$$I(s,\hat{\xi}) = \mu^{-2}\Lambda + \mu^{2}\Lambda^{-1} + \mu^{-2}\eta^{2} . \qquad (7.1)$$

The most important implication of this assumption is this that the resulting one-dimensional strain energy function takes the conventional form

$$\Phi = \Phi(\varepsilon, \eta, \varkappa) . \qquad (7.2)$$

The field equations and boundary conditions for this version of the rod theory can again be obtained by the reduction of the two-dimensional principle of virtual work. However, if this is done exactly we encounter some difficulties. Indeed, the resulting one-dimensional principle of virtual work takes again the form (5.4) only now $\delta\Phi$ is given by

$$\delta\Phi = N d\varepsilon + Q\delta\eta + M\delta\varkappa , \qquad (7.3)$$

where

$$N = \Phi_{\varepsilon}(\mathfrak{e}), \qquad Q = \Phi_{\eta}(\mathfrak{e}), \qquad M = \Phi_{\varkappa}(\mathfrak{e}) , \qquad (7.4)$$

with $\mathfrak{e} = (\varepsilon, \eta, \varkappa)$. The local equilibrium equations and boundary conditions may next be obtained as in chapter 5. Their compoment form would be (5.24) and (5.25), respectively, with $n = m = 0$ in consistence with the form (7.2) of the strain energy function. However, the order of the resulting equilibrium equations precludes to satisfy five boundary conditions (5.25). In order to exclude this inconsistency we must assume that the end foces n^{*} and m^{*} are sufficienty small to be neglected (and this is the additional assumption). If so, then also the distributed forces f and k may also be disregarded. As a result the one-dimensional principle of virtual work consistent with the strain energy function (7.2) is

$$\int_0^1 \delta\Phi ds = \int_0^1 (\mathbf{p}\cdot\delta\bar{\mathbf{r}} + \mathbf{l}\cdot\delta\boldsymbol{\phi})ds + [\mathbf{N}^*\cdot\delta\bar{\mathbf{r}} + \mathbf{M}^*\cdot\delta\boldsymbol{\phi}]_0^1 . \tag{7.5}$$

Then the equilibrium equations and boundary conditions take the form

$$\begin{aligned} &\mathbf{N}' + \mathbf{p} = \mathbf{0} , \\ &\mathbf{M}' + \bar{\mathbf{r}}' \times \mathbf{N} + \mathbf{l} = 0 , \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} &\mathbf{N} = \mathbf{N}^* \quad \text{or} \quad \bar{\mathbf{r}} = \bar{\mathbf{r}}^* , \\ &\mathbf{M} = \mathbf{M}^* \quad \text{or} \quad \boldsymbol{\phi} = \boldsymbol{\phi}^* , \end{aligned} \tag{7.7}$$

or, written in the component form

$$\begin{aligned} &N' - (K - \varkappa)Q + p_1 = 0 , \\ &Q' + (K - \varkappa)N + p_2 = 0 , \\ &M' - (1 + \varepsilon)Q + \eta N + l = 0 , \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} &N = N^* \quad \text{or} \quad (x_1 - x_1^*) \cos\theta + (x_2 - x_2^*) \sin\theta = 0 , \\ &Q = Q^* \quad \text{or} \quad -(x_1 - x_1^*) \sin\theta + (x_2 - x_2^*) \cos\theta = 0 , \\ &M = M^* \quad \text{or} \quad \phi = \phi^* . \end{aligned} \tag{7.9}$$

In this way we arrive at the conventional structure of the rod theory (we call it simple shearable model) developed in [5-7,22] within the direct approach and in [16,17] by the mixed approach based on a weighted reference curve. The complete set of equations for this theory consists of the equilibrium equations (7.8), the constitutive equations (7.4), the kinematical relations (3.24) and the boundary conditions (7.9). Clearly, the independent kinematical variables are $\bar{r}$ (or $\mathbf{u}$) and $\boldsymbol{\phi}$.

In passing it is worthwhile to observe that this version of the rod theory though mathematically consistent involves a physical discrepancy. It lies in this that the transverse normal deformation is suitably accounted for in the determination of the strain energy function and thus constitutive equations but no change in thickness and the associated effects find their reflection in the form of boundary conditions. In other words, no change in thickness can be specified at the rod ends within this version of the theory.

We call the rod theory described by the equations (7.4) - (7.9) simple shearable rod model. These equations may easily be modified to the case when the shear deformation is constrained to vanish, i.e. $\gamma = \eta = 0$. The resulting theory we shall call simple unshearable rod model.

8. Constitutive relations

In the theory under consideration the stress resultants and couples are defined as partial derivatives of the strain energy function with respect to its arguments. We now derive the corresponding formulae. Here we again consider general case, i.e. non-simple shearable rod model.

We define the following quantities

$$\hat{W}_I(I) = \frac{\partial \hat{W}}{\partial I}, \quad \hat{W}_{II}(I) = \frac{\partial^2 \hat{W}}{\partial I^2}, \quad \Xi(I) = \frac{\hat{W}_n}{\hat{W}_I}, \tag{8.1}$$

where it is assumed that $\hat{W}_{II} \neq 0$ for the admissible values of the invariant I. In fact, for real matrials we have $\hat{W}(I) > 0$. This ensures that a material has positive shear modulus [1]. Let further

$$\mathfrak{e} = (\varepsilon, \eta, \varkappa, \varepsilon', \varkappa') . \tag{8.2}$$

Then differentiation of (4.1) yields

$$\Phi_p(\mathfrak{e}, s) = Ch_o \int_{-}^{+} \hat{W}_I I_p \mu d\hat{\xi} , \tag{8.3}$$

where a subscript p indicates partial differentiation with respect to the arguments of Φ, i.e. $p = \varepsilon, \eta, \varkappa, \varepsilon'$ or $\varkappa'$. In the analysis of buckling problems we shall also need second partial derivatives of the strain energy function. From (8.3) we easily obtain

$$\Phi_{pq}(\mathfrak{e},s) = Ch_o \int_{-}^{+} \hat{W}_I(I_{pq} + I_p I_q)\mu d\hat{\xi} . \tag{8.4}$$

According to (4.4) we have

$$I(\mathfrak{e};\hat{\xi}) = \mu^{-2}\Lambda + \mu^2\Lambda^{-1} + \mu^{-2}(\eta + v)^2 , \tag{8.5}$$

where $v \equiv \zeta'$. Assuming now that $K \neq 0$ from (4.7) one gets

$$v(\varepsilon, \varkappa, \varepsilon', \varkappa'; \hat{\xi}) = -(a\varepsilon' + b\varkappa' + cK'),$$

$$\begin{aligned} a(\varepsilon, \varkappa; \hat{\xi}) &= \frac{\zeta}{\sqrt{\Lambda}} . \\ b(\varepsilon, \varkappa; \hat{\xi}) &= \frac{\zeta^2}{2\sqrt{\Lambda}} . \\ c(\varepsilon, \varkappa; \hat{\xi}) &= \frac{1}{\tilde{K}} \{\zeta + \frac{1}{2K^2\sqrt{\Lambda}} [K(\mu^2 - 1) + \tilde{K}(\mu - 1)^2]\} , \end{aligned} \tag{8.6}$$

with Λ defined by (4.5) and

$$\tilde{K} = K - \varkappa = -(\epsilon_h + h_o\varkappa), \quad K = -h_o^{-1}\epsilon_h . \tag{8.7}$$

Next, from (8.5), (8.6) and (4.5) we readily find

$$\begin{aligned} I_\varepsilon &= 2\mu^{-1}[(1+\varepsilon)G + \mu^{-1}(\eta + v)v_\varepsilon] , \\ I_\eta &= 2\mu^2(\eta + v) \\ I_\varkappa &= \mu^{-1}[-K^{-1}G(\mu^2 - 1) + 2\mu^{-1}(\eta + v)v_\varkappa] , \\ I_{\varepsilon'} &= -2\mu^{-2}a(\eta + v) , \\ I_{\varkappa'} &= -2\mu^{-2}b(\eta + v) , \end{aligned} \tag{8.8}$$

where

$$G = \mu^{-1} - \mu^3\Lambda^{-2} . \tag{8.9}$$

Subsequent differentiation of (8.8) yields

$$I_{\varepsilon\varepsilon} = 2\mu^{-1}\{G + 4(1+\varepsilon)^2\mu^3\Lambda^{-3} + \mu^{-1}[v_\varepsilon^2 + (\eta + v)v_{\varepsilon\varepsilon}]\} ,$$

$$I_{\varepsilon\eta} = 2\mu^{-1}v_\varepsilon ,$$

$$I_{\varepsilon\varkappa} = 2\mu^{-1}\{-2(1+\varepsilon)K^{-1}\mu^3\Lambda^{-3}(\mu^{-2} - 1) + \mu^{-1}[v_\varepsilon v_\varkappa + (\eta + v)v_{\varepsilon\varkappa}]\} ,$$

$$I_{\varepsilon\varepsilon'} = -2\mu^{-2}[av_\varepsilon + a_\varepsilon(\eta + v)] ,$$

$$I_{\varepsilon\varkappa'} = - 2\mu^{-2} [av_{\varepsilon} + b_{\varepsilon}(\eta + v)] ,$$

$$I_{\eta\eta} = 2\mu^{-2} .$$

$$I_{\eta\varkappa} = 0 ,$$

$$I_{\eta\varepsilon'} = - 2\mu^{-2}a , \tag{8.10}$$

$$I_{\eta\varkappa'} = - 2\mu^{-2}b ,$$

$$I_{\varkappa\varkappa} = 2\mu^{-1} \{K^{-2}\mu^{3}\Lambda^{-3}(\mu^{2} - 1)^{2} + \mu^{-1}[v_{\varkappa}^{2} + (\eta + v)v_{\varkappa\varkappa}]\} ,$$

$$I_{\varkappa\varepsilon'} = - 2\mu^{-2} [av_{\varkappa} + a_{\varkappa}(\eta + v)] ,$$

$$I_{\varkappa\varkappa'} = - 2\mu^{-2} [bv_{\varkappa} + b_{\varkappa}(\eta + v)] ,$$

$$I_{\varepsilon'\varepsilon'} = 2\mu^{-2}a^{2} ,$$

$$I_{\varepsilon'\varkappa'} = 2\mu^{-2}ab ,$$

$$I_{\varkappa'\varkappa'} = 2\mu^{-2}b^{2} .$$

The formulae (8.1) - (8.10) provide explicit form of the constitutive relations for the rod made of an arbitrary hyperelastic material. We emphasize that the constitutive equations derived above are exact within the assumptions I) and II).

The formulae (8.1) - (8.10) may easily be reduced to the special cases of the general theory discussed in Chapters 6 and 7 as well as to the case of the iniatially straight rod, i.e. when $K = 0$.

9. Buckling of straight rods

9.1 Formulation of the problem

Consider a straight rod compressed along its axis by forces P applied through rigid plates at the rod ends (Fig. 2). The plates are assumed to be well-lubricated and constrained such that they remain perpendicular to the axis. Let a material of the rod be hyperelastic, isotropic, and incompressible so that its mechanical properties are determined by a strain energy density (2.13). In the three-dimensional sense the rod we are concerned with here is to be viewed as the infinite rectangular strip. This is consistent with our interpretation of the planar deformation of rods.

For sufficiently small values of the forces P the deformation of the rod will be homogenous with the rod axis remaining straight. The strains in this state are uniquely determined by the axial stretch $\lambda = L/L_o$, where L_o and L denote the initial and current length of the rod. The buckling problem is then formulated as: determine the critical values P_c of the load and the critical stretches λ_c for which the deformation ceases to be homogeneous. This problem has been studied by many authors within the three-dimensional finite elasticity under the plane-strain assumption (cf. [14,20,21]). It was shown there that the buckling phenomena is not necessarity associated with the slenderness of the rod and may occur in very thick one. Moreover, the asymmetric (flexural) as well as symmetric (bulging) instability has been observed.

In this chapter we reconsider the problem employing the rod theory developed in this paper. The analysis is carried out within general theory (non-simple shearable rod model) as well as within particular cases of this theory. In this way we shall illustrate the significance of various implifications usually made in the derivation of the basic rod equations.

9.2 Non-simple shearable model

In the absence of the distributed loads the equilibrium equations (5.24) take the form

$$(N - n')' + \varkappa Q = 0 \; ,$$

$$Q' - \varkappa(N - n') = 0 \; , \qquad (9.1)$$

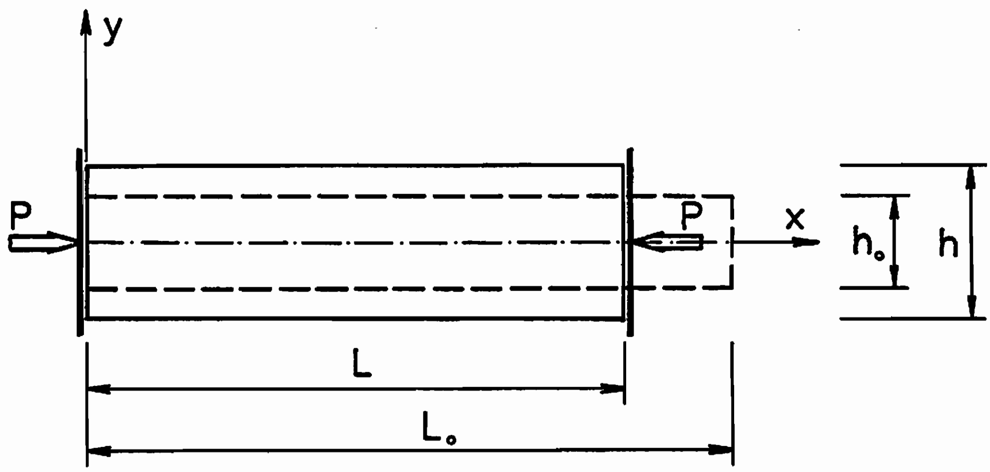

Fig. 2 Straight rod under axial compression, initial and pre-buckling state

$$(M - m')' - (1 + \varepsilon)Q + \eta(N - n') = 0 \ .$$

Moreover, since the rod may freely expand in the direction normal to the axis the associated boundary conditions are

$$N - n' = -P, \quad Q = \phi = n = m = 0, \quad s = 0,L \ . \tag{9.2}$$

The equilibrium equations (9.1) and the boundary conditions (9.2) supplemented by the constitutive equations (5.14), (5.15) and the kinematical relations (3.24) constitute the non-linear boundary value problem to be analized. This boundary value problem admit a trivial solution $\varepsilon_o = \lambda - 1$, $\gamma_o = \phi_o = 0$ and hence $\eta_o = \varkappa_o = \varepsilon'_o = \varkappa'_o = 0$ corresponding to a straight unsheared state of the rod. With $\mathfrak{e}$ defined by (8.2) we shall denote the trivial solution by $\mathfrak{e}_o = (\lambda - 1, 0, 0, 0, 0)$. Moreover, we take the reference curve of the rod to be the middle one so that $\hat{\xi} \in [-\frac{1}{2}, +\frac{1}{2}]$. Then from (8.5), (8.8), (8.3) and (5.14), (5.15) we have

$$I_o(\lambda) \equiv I(\hat{\xi}; \mathfrak{e}_o) = \lambda^2 + \lambda^{-2} \ , \tag{9.3}$$

and

$$N^o = \Phi^o_\varepsilon(\lambda) = \Phi_\varepsilon(\mathfrak{e}_o) = 2Ch_o\hat{W}^o_I(\lambda - \lambda^{-3}) \ , \tag{9.4}$$

$$Q^o = M^o = n^o = m^o = 0 \ , \tag{9.5}$$

where

$$\hat{W}^o_I = \hat{W}^o_I(\lambda) \equiv \hat{W}_I(I_o) \tag{9.6}$$

Introducing now (9.4) and (9.5) into (9.1) and making use of the boundary conditions (9.2) we find out that the equilibrium equations reduce to the single algebraic equation for the axial stretch λ

$$\Phi^o_\varepsilon(\lambda) + P = 0 \ . \tag{9.7}$$

This equation may be rewritten in the form

$$\hat{W}^o_I(\lambda - \lambda^{-3}) + P^* = 0 \ , \quad P^* \equiv \frac{P}{2Ch_o} \ . \tag{9.8}$$

Moreover, from (4.9) we easily obtain that the initial and current rod thickness are related by $h = \lambda^{-1}h_o$. It is to be noted that the obtained trivial solution contains no approximation and thus it is exact in the sense of the three-dimensional finite elasticity (cf. [14,20]).

In order to determine possible buckling states of the rod we now set

$$\varepsilon = \varepsilon_o + \varepsilon_1 \ . \quad (\eta,\ \varkappa,\ \varepsilon',\ \varkappa') = (\eta_1 .\ \varkappa_1 .\ \varepsilon_1' .\ \varkappa_1') \ , \tag{9.9}$$

$$N = N^o + N_1 \ . \ (Q,\ M,\ n,\ m) = (Q_1 .\ M_1 .\ n_1 .\ m_1) \ ,$$

where $\varepsilon_1(s),\dots,m_1(s)$ denote the increments of the respective quantities that take the rod from the trivial solution to the adjacent equilibrium state (they can be taken as small as we wish). Substitution of (9.9) into (9.1) and subsequent linearization about the trivial solution yields the following buckling equations

$$(N_1 - n_1')' = 0 \ ,$$

$$Q_1' - \varkappa_1 N^o = 0 \ , \tag{9.10}$$

$$(M_1 - m_1')' - \lambda Q_1 + \eta_1 N^o = 0 \ .$$

The corresponding linearized boundary conditions obtained from (9.2) are

$$N_1 - n_1' = 0 \ , \quad Q_1 = \phi_1 = n_1 = m_1 = 0 \quad \text{at} \quad s = 0,L \ . \tag{9.11}$$

Furthermore, straightforward though rather lenghy calculations with the use of (8.4) - (8.10) and employing de l'Hospital rule yield

$$\Phi^o_{\varepsilon\varepsilon} = 2Ch_o\, \hat{W}^o_I\, [1 + 3\lambda^{-4} + 2\, \Xi^o(\lambda - \lambda^{-3})^2] \ ,$$

$$\Phi^o_{\eta\eta} = 2Ch_o\, \hat{W}^o_I \ .$$

$$\Phi^o_{\varkappa\varkappa} = \frac{2}{3}\, Ch_o^3\, \hat{W}^o_I\, [\lambda^{-6} + \frac{1}{2}\, \Xi^o(1 - \lambda^{-4})^2] \ , \tag{9.12}$$

$$\Phi^o_{\varepsilon'\varepsilon'} = \frac{1}{6}\, Ch_o^3 \hat{W}^o_I\, \lambda^{-4} \ ,$$

$$\Phi^o_{\varkappa'\varkappa'} = \frac{1}{160} Ch_o^5 \hat{W}_I^o \lambda^{-6} ,$$

$$\Phi^o_{\eta\varkappa'} = -\frac{1}{12} Ch_o^3 \hat{W}_I^o \lambda^{-3} ,$$

with remaining second partial derivatives of Φ vanishing at the trivial solution. Here $\Phi^o_{pq} = \Phi^{pq}(\ _o)$ and $\Xi^o = \Xi(I_o)$, where Ξ is defined by (8.1). According to (9.12) the linearized constitutive relations about the trivial solution are given by

$$N_1 = \Phi^o_{\varepsilon\varepsilon}\, \varepsilon_1 \,.\quad Q_1 = \Phi^o_{\eta\eta}\, \eta_1 + \Phi^o_{\eta\varkappa'}\, \varkappa_1' \,.\quad M_1 = \Phi^o_{\varkappa\varkappa}\, \varkappa_1 \,.$$

$$n_1 = \Phi^o_{\varepsilon'\varepsilon'}\, \varepsilon_1' \,.\quad m_1 = \Phi^o_{\varkappa'\eta}\, \eta_1 + \Phi^o_{\varkappa'\varkappa'}\, \varkappa_1' \,. \qquad (9.13)$$

In turn, the linearization of the kinematical relations (3.24) yields

$$\varepsilon_1(s) = u_1' .\quad \eta_1(s) = w_1' + \lambda\phi_1 .\quad \varkappa_1(s) = \phi_1' \,. \qquad (9.14)$$

In consequence the linearized equilibrium equations (9.10) and the associated boundary conditions (9.11) reduce to the form

$$N_1 - n_1' = 0 ,$$

$$Q_1 - \Phi^o_\varepsilon\, \phi_1 = 0 , \qquad s \in [0, L_o] \qquad (9.15)$$

$$M_1' - m_1'' - \lambda Q_1 + \Phi^o_\varepsilon\, \eta_1 = 0 ,$$

$$Q_1 = \phi_1 = n_1 = m_1 = 0 \qquad \text{at } s = 0, L_o \qquad (9.16)$$

Finally, upon making use of the constitutive equations (9.13) the buckling problem (9.15), (9.16) may be reduced to two separate nonlinear eigenvalue problems for the axial stretch λ

$$\varepsilon_1'' - h_o^{-2} f(\lambda)\varepsilon_1 = 0 , \qquad s \in [0, L_o]$$

$$\varepsilon_1'(0) = \varepsilon_1'(L_o) = 0 , \qquad (9.17)$$

and

$$\phi_1^{IV} - h_o^{-2}\, k(\lambda)\phi_1'' - h_o^{-4}\, h(\lambda)\phi_1 = 0 \ , \qquad s \quad [0,\, L_o]$$

$$\phi_1(0) = \phi_1(L_o) = \phi_1''(0) = \phi_1''(L_o) = 0 \ , \tag{9.18}$$

where

$$f(\lambda) = 12[3 + \lambda^4 + 2\lambda^{-2} \quad {}^{o}(1 - \lambda^4)^2] \ , \tag{9.19}$$

$$k(\lambda) = 5\, f(\lambda) = 60[3 + \lambda^4 + 2\lambda^{-2} \quad {}^{o}(1 - \lambda^4)^2] \ , \tag{9.20}$$

$$h(\lambda) = 720(1 - \lambda^4) \tag{9.21}$$

A number λ_c for which (9.18), respectively (9.17), has non-trivial solution is an eigenvalue (critical stretch) of the problem and the corresponding solution is an eigenfunction (buckling mode). We note that once the critical stretch is determined the corresponding critical force P_c may easily be found from (9.8). The existance of the non-trivial solutions to the problems (9.17) and (9.18) critically depends on the behavior of a material the rod is made of. In order to ensure that a material is well-behaved physically we adapt the ellipticity condition as a constitutive restriction (see [1] for the implications of loss of ellipticity). Abeyaratne [1] has show that for the plane strain problem

$$1 + 2(I - 2)\,\Xi(I) > 0 \ , \tag{9.20}$$

is the necessary and sufficient condition for ellipticity (as before we also assume that $\hat{W}_I(I) > 0$). From $(9.12)_1$ and (9.3) we find that the ellipticity condition (9.20) implies that $\Phi^{o}_{\varepsilon\varepsilon}(\lambda) > 0$. Thus $\Phi^{o}_{\varepsilon}(\lambda)$ is the monotonous increasing function of the axial stretch λ. In consequence of this result we find that $\lambda < 1$ for the compression of the rod and $\lambda > 1$ for its tension. Furthermore, the ellipticity condition (9.20) implies that $f(\lambda)$ defined by (9.19) is positive, $f(\lambda) > 0$. Consequently the only solution of the problem (9.17) is the trivial solution $\varepsilon_1(s) = 0$. Hence there are no eigenvalues of this problem.

In turn, for the compression of the rod, i.e. for $\lambda < 1$, the ellipticity condition (9.20) implies that λ_c is an eigenvalue of the problem (9.18) if

$$\sqrt{k^2(\lambda_c) + 4\, h(\lambda_c)} - k(\lambda_c) = 2n^2\pi^2\left(\frac{h_o}{L_o}\right)^2 \ , \tag{9.21}$$

for $n = 1,2,\ldots$. The corresponding eigenfunctions (flexural buckling modes) are

$\phi_1(s) = \sin (\frac{n\pi}{L_o})$.

The results of a numerical solution of (9.21) are shown in Fig. 3 and 4. The comparison with the three-dimensional solution [20] shows that the rod theory developed in this paper provides correct solution to the problem even for relatively thick rod. However, there exists fundamental divergence in the solutions when the slenderness parameter L_o/h_o tends to zero, i.e. when the rod becomes a thin wafer. Also, the rod theory does not reveal the existance of the symmetric type of buckling. Moreover, as it is evident from Fig. 4 the experimental results [8] show generally poor agreement with either the three-dimensional or rod theory solutions whenever L_o/h_o is smaller than 2.0 .

9.3 Alternative rod models

Following the same way we can now obtain the solutions to the problem employing particular versions of the general rod theory discussed in Chapters 6 and 7. The representative results are shown in Fig. 5. As it is seen from this Fig. the shear deformation has great influence on the buckling behaviour of rods.

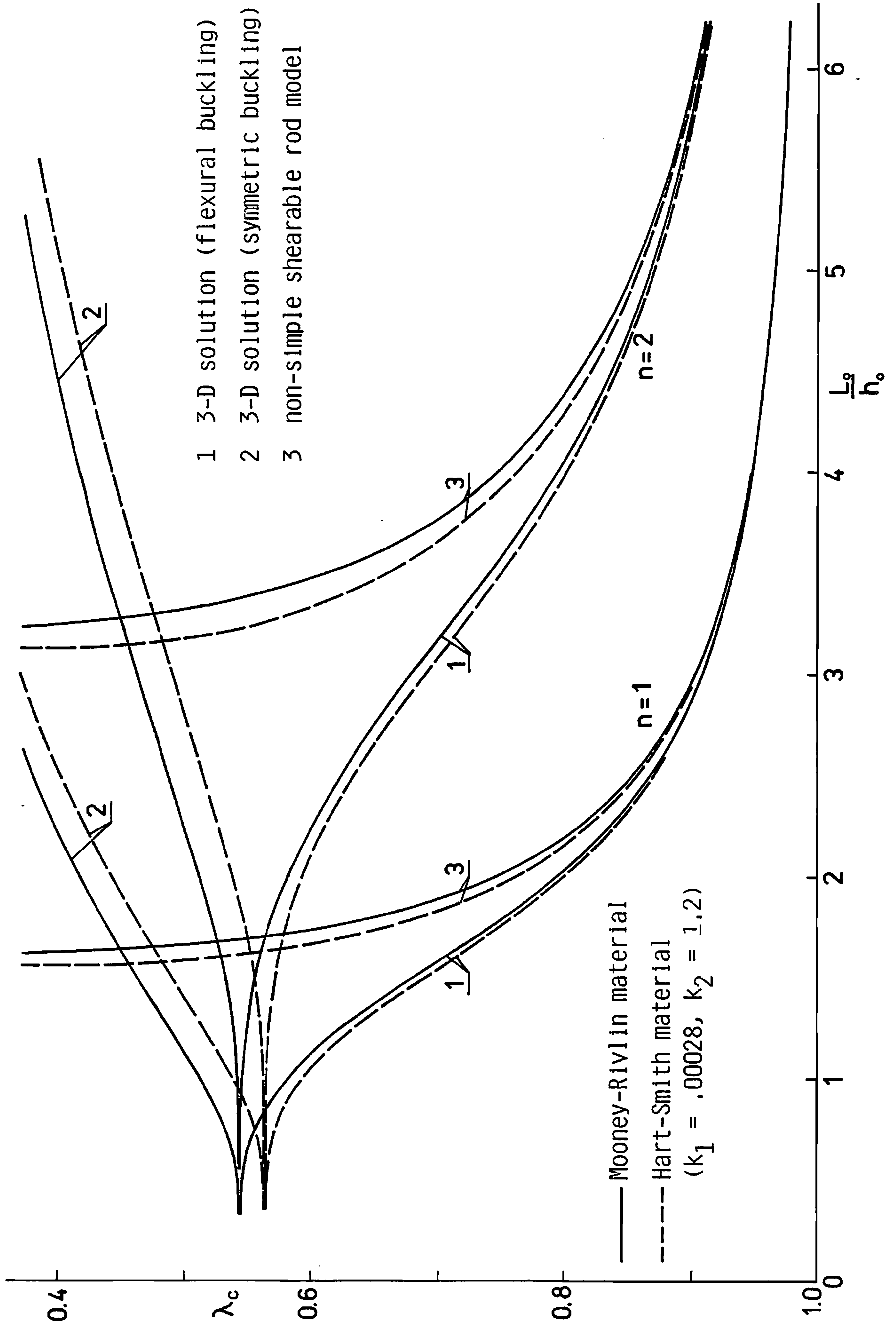

Fig. 3 Straight rod under axial compression, critical stretch vs thickness

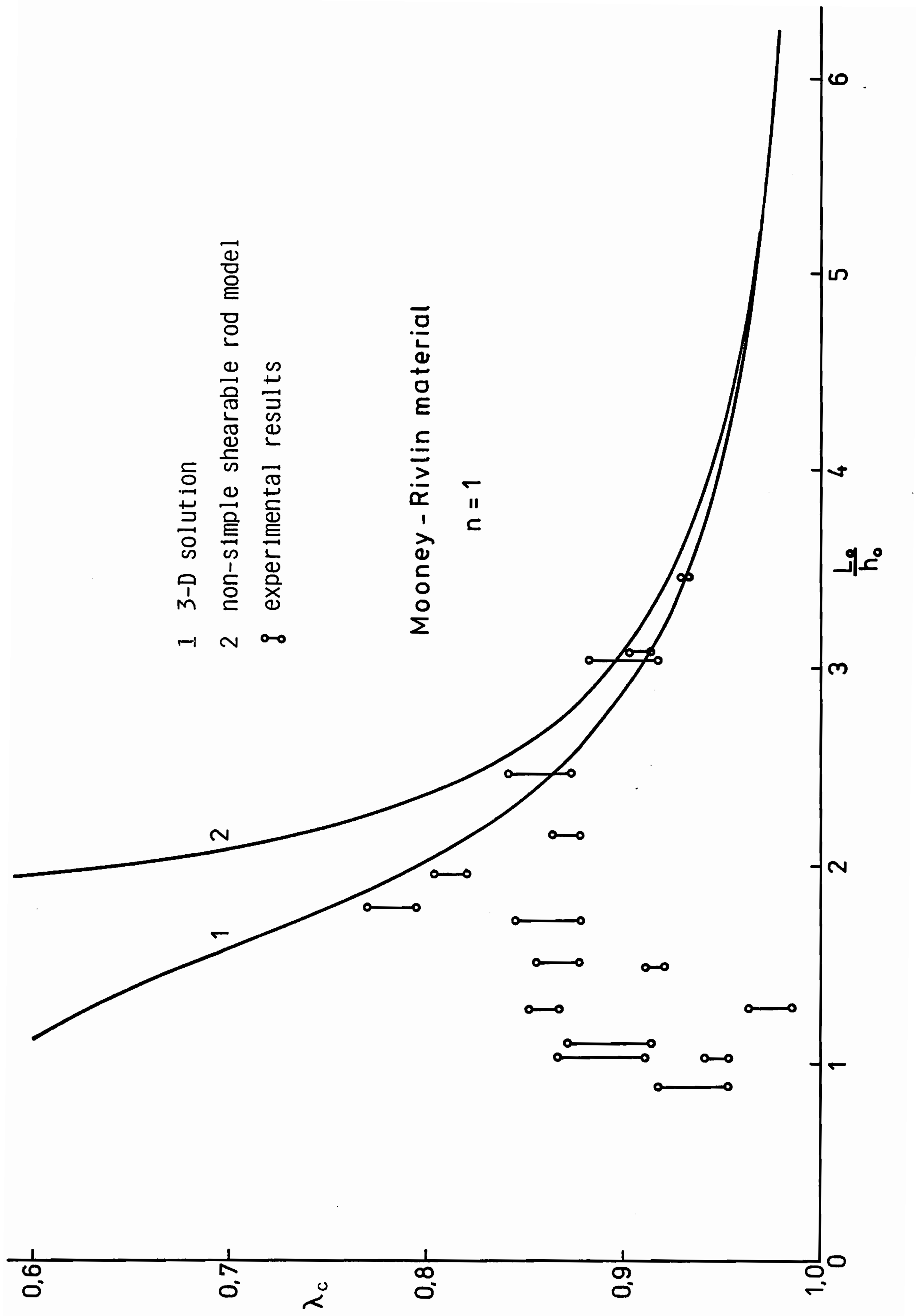

Fig. 4 Straight rod under axial compression, comparison of solutions and experimental results

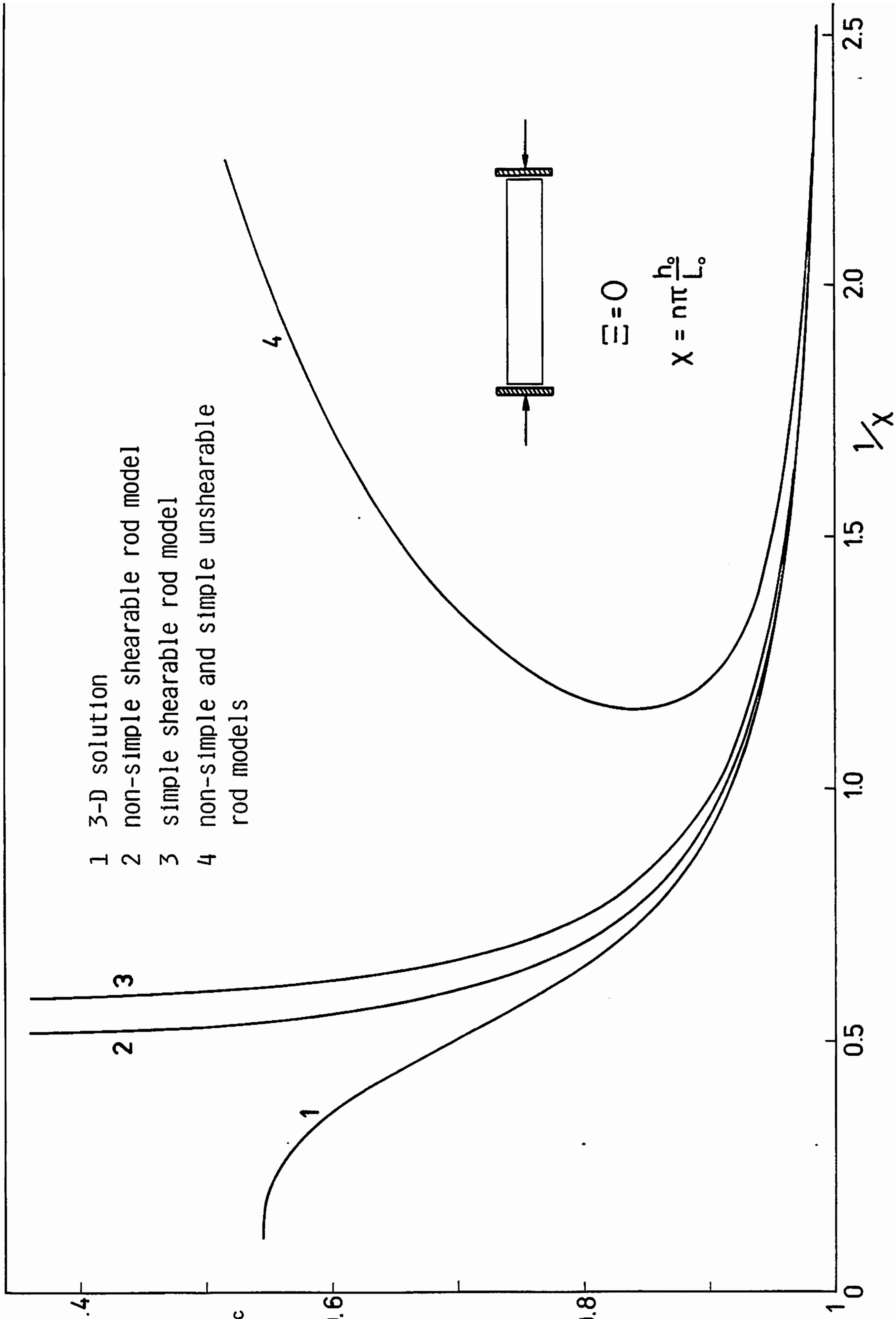

Fig. 5 Straight rod under axial compression, solutions predicted by different versions of the rod theory

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