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# Variational principles of fracture mechanics

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Summary

This report presents the principle of total energy for statics of a geometrically nonlinear elastic body whose initial configuration contains a gap. From this principle statical equations and boundary conditions for the body with the gap are derived by calculating the variation of the energy functional on a set of admissible configurations which is not a linear space. The equilibrium condition at a gap tip is associated with the well-known J-integrals. The method of generalization of this variational principle to quasistatics of a geometrically nonlinear elastic-plastic body with a crack is developed by introducing the concept of internal degrees of freedom. As a result the condition of gap fixation or crack propagation for an elastic-plastic body is obtained. By including the reaction of inertia the principle of total energy is transformed to the variational inequality of evolution expressing the principle of virtual work. However, it is shown by analysis of the equation of energy balance that in the inequality of evolution the flux of kinetic energy entering into the crack tip must be taken into account. By combining the consequences of the inequality of evolution and the equation of energy balance a closed system of dynamical equations, boundary conditions and additional conditions on the unknown contact crack surfaces and crack tip for both kinds of elastic and elastic-plastic bodies is obtained. The variants of the geometrically linear theory of fracture mechanics are considered. Finally, to illustrate the theory, the solutions of two statical problems are presented.

Contents

## 1. Introduction.

Let us consider a deformable body whose initial configuration contains a gap. By a gap we mean a two-dimensional surface of material discontinuity which is obtained as a result of a structural defect or a deformation history. Let this body be deformed by external forces and tractions. Under which condition of loadings will the gapped body be in the state of equilibrium, and when does the gap become the crack propagating throughout the body? The aim of fracture mechanics is to construct a system of basic equations governing such a response of a gapped body to applied external forces and tractions.

In the present report we shall follow this aim starting from the principle of total energy. For statics of a geometrically nonlinear elastic body this principle states: an equilibrium criterion for a geometrically nonlinear elastic body with a gap is that the variation of its total energy at the actual configuration is non-negative for an arbitrary family of admissible comparison configurations . The peculiarity of this variational principle is that it allows for comparison configurations to have surfaces of discontinuity containing the initial gap but differing from it.

If the body does not contain a gap, then the above stated variational principle is reduced to the classical principle of stationary energy (Kirchhoff, Gibbs [1,2]; see also the recent papers in [3-5]). Gibbs also presented interesting applications of the energy principle to other theories, such as the theory of heterogeneous media, the theory of capillarity, and the theory of phase transition. The first step of generalization of the principle of stationary energy to fracture mechanics was made in the pioneer works of Griffith [6-7] . For determining a critical size of an equilibrium gap (in a case of plane deformation) Griffith differentiated the total body energy expressed in terms of the gap size and equated it to zero. This fundamental idea has then been developed into the theory of brittle fracture by various authors (Irwin, Orowan, Cherepanov, Rice,and others [8-16]). The fact that all basic equations for an elastic body with a gap, including the equilibrium condition at the gap tip, can be deduced from the principle of total energy as a variational inequality was established in [17-19]. This variational inequality is non-classical in the sense that the set of admissible configurations, containing functions with one-side restrictions and different regions of definition, is not a linear space. It is possible, nevertheless, to introduce the concept of continuity and differentiability of the energy functional on this set of admissible configurations and to calculate its variation. Due to the singularity of the deformation gradients stipulating also the singularity of the stress field at the gap tip [20-23], a non-zero vector of flux of energy entering into the gap tip during the deformation process is formed. This flux of energy can be calculated by J-integrals, the origin of which can be found in Eshelby's work [24]. With the help of the J-integrals Cherepanov and Rice have derived different variants of the equilibrium condition at the gap tip [10,11]. The gap tip condition, obtained from the variational principle of total energy, in the case of geometrically linear theory coincides with that of Cherepanov [17-18] and has the following physical sense: in the equilibrium configuration the module of the flux of energy entering into the gap tip should be less than, or equal to, the double surface energy density.

For a real material a plastic deformation occurs when the stress attains some surface in a stress space. Consequently, the stress singularity near the gap tip described by the elasticity theory is never realized, and for obtaining the more adequate local distribution of stress and deformation gradients a model of an elastic-plastic body has to be involved. By an elastic-plastic body we mean a body whose mechanical state includes an additional measure of plastic deformation, refered to as internal degrees of freedom [3,25-27] These internal degrees of freedom should satisfy the yield condition and the nonholonomical generalized associate law [27-32]. For the external degrees of freedom - the actual configuration of the body- the following variational principle is valid: the variation of the energy functional of a geometrically nonlinear elastic-plastic body taken at the actual configuration relative to arbitrary family of admissible comparison configurations vanishes for every time. The quasistatical equations derived from this principle, the yield condition, and the associate law compose the closed system of equations for determining all the unknown functions of the theory. It is shown that the variational equation will be transformed to the equation of energy balance if an arbitrary family of admissible configurations is replaced by the real motion of the body. Generalizing this variational principle to quasistatics of an elastic-plastic body with a crack in a manner analogical to that which was used before in the elasticity theory, we obtain the mathematical formulation of the boundary-value crack problem. The condition at the crack tip is associated again with the J-integrals, but now the contour of integration should lie in the plastic zone. Because of the yield condition there is a redistribution of the stress gradients near the crack tip [33-37]. Therefore, the J-integrals depend upon characteristics of this redistribution. For an elastic-perfectly-plastic body with a crack the J-integrals, generally speaking, are not defined [38]. However, if we consider this body as a limit case of an elastic-plastic body with hardening coefficient tending toward zero, the limit values of the J-integrals can be well-defined. The condition at the crack tip associated with these integrals is distinguished from the well-known $K_{1c}$, J , $\delta$ and other criteria (see [39-43]).

If the statical or quasistatical crack problem has no stable solutions the crack will quickly propagate throughout the body and the account of dynamical factors is then necessary. This account is non-trivial and can be made within the framework of a variational inequality of evolution (Lion, Stampachia, Duvaut [44-46]). It will be shown that in the variational inequality of fracture dynamics one must include both the usual d'Alembert's force of inertia, and the flux of kinetic energy entering into a crack tip [17-19]. This fact follows from analysis of the equation of energy balance. Combining consequences of the inequality of evolution and the equation of energy balance, one obtains the closed system of dynamical equations, boundary conditions and additional conditions on the unknown contact surfaces and crack tip for both kinds of elastic and elastic-plastic bodies. The dynamical condition at a crack tip, associated with I-integrals [17-19], determines the location as well as the direction of propagation of the crack.

We shall also consider the simple variants of the geometrically linear theory of fracture and their variational formulation. Particular attention is focused on the fact that here the crack problem as a whole remains nonlinear. This fact plays a very important role in the application of the method of homogenization [47-48].

To illustrate the constructed theory we consider two examples: a) a statical problem of a neo-Hookean incompressible infinite slab containing a gap under a simple shear at infinity [49], and b) a statical plane problem of a geometrically linear elastic slab containing an angled gap under a tensile stress at infinity . In both problems the calculation of J-integrals is available and the equilibrium condition at a gap tip is written in an explicit form.

## 2. Statics of a geometrically nonlinear elastic body with a gap.

Let an initial configuration of a geometrically nonlinear elastic body contain a gap which is modeled by a smooth surface $\Omega$ with a smooth boundary $\partial\Omega$. This configuration, occupying the region $V_\Omega = V\setminus(\Omega\cup\partial\Omega)$ of Euclidean three-dimensional space with the exterior boundary $\partial V$ and the interior boundary $\Omega\cup\partial\Omega$, is chosen to be a reference configuration. The Cartesian co-ordinates of a typical point of the initial configuration is denoted by $X_a$, $a=1,2,3$. In a deformed configuration this material point has new co-ordinates $x_i$ given by

$$x_i = x_i(X_1, X_2, X_3) \ , \ i=1,2,3, \ x_i \in v \tag{2.1}$$

The co-ordinates $x_i$ run over the region v of the deformed configuration. If the deformed body is in an equilibrium state, then the functions $x_i(X_a)$ perform a one-to-one continuously-differentiable transformation of $V_\Omega$ into v, satisfying the condition

$$0 < \det\left|\frac{\partial x_i}{\partial X_a}\right| < \infty \quad , \quad \forall\ X_a \in V_\Omega \tag{2.2}$$

At points $X_a \in \Omega\cup\partial\Omega$ the functions $x_i$ need not be defined. The limit values (the traces) of $x_i$ on two sides of $\Omega$ are denoted by $x_i^+$ and $x_i^-$ . They describe two surfaces of the gap in the deformed configuration (Fig.1). It is obvious that $x_i^+$ and $x_i^-$ may differ from each other. In other words the functions $x_i(X_a)$ may have the jump on $\Omega$. Therefore we shall sometimes call the surface $\Omega$ a singular surface of the configuration $x_i(X_a)$. We shall suppose, furthermore, that the deformation gradients $\partial x_i/\partial X_a$ may have a singularity at points of $\partial\Omega$ in the sense that $\partial x_i/\partial X_a \Rightarrow \infty$ when $X_a \Rightarrow \partial\Omega$. This kind of singularity is due to purely geometrical factors and may occur in bodies with different physical behaviour.

According to the variational principle of total energy stated in the Introduction, the mathematical formulation of a statical problem for a body with a gap requires three definitions: a) a set of admissible comparison configurations, b) a total energy functional on this set, and c) a variation of this functional. These definitions will be given in the following paragraph.

The set $\mathcal{E}$ of admissible configurations, compared with an equilibrium configuration, consists of all one-to-one continuous piecewise differentiable transformations $y_i(X_a)$ of regions $V_\Sigma=V\setminus(\Sigma\cup\partial\Sigma)$ into open regions of the Euclidean space, where $\Sigma$ is an arbitrary two-dimensional surface containing $\Omega$. The continuity and smoothness of $\Sigma$ are assumed, except at points of $\partial\Omega$, where the surface $\Sigma$ may, generally speaking, have a non-smooth continuation of $\Omega$ (Fig.2). This assumption is adopted here in order to compare the energy among configurations with singular surfaces continuing $\Omega$ along all non-tangential directions. It is also assumed that the deformation gradients $\partial y_i/\partial X_a$ may have singularities at points of $\partial\Sigma$. The admissible configurations $y_i(X_a)$ should also satisfy the kinematical condition $y_i(X_a)=r_i(X_a)$ on that part $\partial V_x$ of the exterior boundary, where kinematical boundary conditions are prescribed.

By generalizing Griffith's idea [6-7], we postulate the following expression for the total energy functional of a homogeneous elastic body with a gap on arbitrary configuration $y_i(X_a)$ with a surface of discontinuity $\Sigma$

$$\mathcal{E}[y_i(X_a)] = \int_{V_\Sigma} \rho_0 f(y_{i,a},\vartheta)\, dX + \int_\Sigma 2\gamma\, dA + \int_{V_\Sigma} \rho_0\Phi(y_i)\, dX - \int_{\partial V_T} T_i y_i dA \qquad (2.3)$$

In the formula (2.3) $\rho_0$ denotes the mass density of the material in its initial state, $f(y_{i,a},\vartheta)$ and $\gamma$ are the free energy per unit mass and the surface energy per unit area respectively, and $\Phi(y_i)$ is the potential of the mass force. The tensor $y_{i,a}= \partial y_i/\partial X_a$ corresponds to the deformation gradients, while $\vartheta$ is the given temperature. The temperature remains constant, therefore, it is not necessary to list it among the variables of the function f.On the remaining part $\partial V_T=\partial V\setminus\partial V_x$ of the exterior boundary $\partial V$ the "dead" traction $T_i$ is prescribed . Throughout the text the Latin indices run from 1 to 3 , the comma is used to denote partial differentiation with respect to $X_a$ and the repeated suffix is used to denote summation.

For the definition of variation let us consider a one-parameter family of admissible configuration $y_i=y_i(X_a,\varepsilon)$ continuously depending on $\varepsilon$ with surfaces of discontinuity $\Omega^\varepsilon$ (a generalized curve in the set $\mathscr{C}$ of admissible configurations). We shall suppose that

$$\Omega^{\varepsilon'} \supseteq \Omega^\varepsilon \supseteq \Omega \quad \text{for } \varepsilon'>\varepsilon>0, \quad \Omega^\varepsilon \Rightarrow \Omega \quad \text{when } \varepsilon\Rightarrow 0$$

$$y_i(X_a,0) = x_i(X_a), \quad y_i(X_a,\varepsilon)= r_i(X_a) \quad \text{for } X_a\in \partial V_x$$

Then the variation of the functional (2.3) can be defined as follows

$$\delta\mathscr{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathscr{E}[y_i(X_a,\varepsilon)] \tag{2.4}$$

According to the principle of total energy, the variation $\delta\mathscr{E}$ should be non-negative for all families of admissible configurations if the actual configuration $x_i(X_a)$ is in a state of equilibrium [17-19]

$$\delta\mathscr{E} \geq 0 , \quad \forall\ y_i(X_a,\varepsilon)\in \mathscr{C} \tag{2.5}$$

It should be noted that the variational problem (2.5) is non-classical in the sense that the set $\mathscr{C}$ of admissible configurations is not a linear space. In fact, due to the impenetrability condition (2.2) and the different regions of definition, one can not add two admissible configurations to obtain a new one.

In order to derive consequences from (2.5) one must calculate $\delta\mathscr{E}$ according to the definition (2.4). The difficulty of this calculation is associated with the changable regions $V_{\Omega^\varepsilon}$ and surfaces $\Omega^\varepsilon$ of integration. To fix these regions and surfaces we shall introduce a family of one-to-one continuous piecewise differentiable autotransformations of V into itself according to a rule $Y_a=Y_a(X_a,\varepsilon)$ so that here

$$V_\Omega \overset{Y_a}{\Rightarrow} V_{\Omega^\varepsilon} \quad , \quad \Omega \overset{Y_a}{\Rightarrow} \Omega^\varepsilon$$

$$Y_a(X_a,\varepsilon) = X_a \quad \text{when } \varepsilon=0 \text{ or } X_a\in \partial V$$

As a kind of "change of variables", $Y_a(X_a,\varepsilon)$ will conveniently be called a parametrization of medium. By using this parametrization, one can calculate the variation of free energy

$$\delta \int\limits_{V_{\Omega}\varepsilon} \rho_0 f(y_{i,a})dX = \delta \int\limits_{V_{\Omega}} \rho_0 f\left(\frac{\partial y_i}{\partial Y_a}\right)\det\left|\frac{\partial Y_a}{\partial X_b}\right| dX =$$

$$= \int\limits_{V_{\Omega}} \left[ \rho_0 \delta f + \rho_0 f \ \delta \det\left|\frac{\partial Y_a}{\partial X_b}\right| \right] dX =$$

$$= \int\limits_{V_{\Omega}} \left[ T_{ia} \ \delta \left(\frac{\partial y_i}{\partial Y_a}\right) + \rho_0 f \ (\delta Y_a)_{,a} \right] dX$$

Here and henceforth the symbol $\delta$ under the integral sign is used to denote partial differentiation with respect to $\varepsilon$ for fixed $X_a$ and $\varepsilon=0$. For example

$$\delta Y_a = \frac{\partial}{\partial \varepsilon}\Big|_{X_a=\mathrm{const},\,\varepsilon=0} Y_a(X_a,\varepsilon)$$

The first stress tensor of Piola-Kirchhoff is given by the formula

$$T_{ia} = \rho_0 \frac{\partial f}{\partial x_{i,a}}$$

It is easy to show, that

$$\delta \left(\frac{\partial y_i}{\partial Y_a}\right) = \delta y_{i,a} - x_{i,b} \ (\delta Y_b)_{,a}$$

$$\delta y_i = \frac{\partial}{\partial \varepsilon}\Big|_{X_a=\mathrm{const},\,\varepsilon=0} y_i(Y_a(X_a,\varepsilon),\varepsilon)$$

Therefore

$$\delta \int\limits_{V_{\Omega}\varepsilon} \rho_0 f \ dX = \int\limits_{V_{\Omega}} ( T_{ia}\delta y_{i,a} + \mu_{ab}\delta Y_{a,b} ) \ dX \tag{2.6}$$

where the tensor

$$\mu_{ab} = -T_{ib}x_{i,a} + \rho_0 f \ \delta_{ab}$$

is the tensor of Eshelby [24], which closely resembles the tensor of chemical potential in the theory of phase transition [53-54], and $\delta_{ab}$ is the Kronecker symbol. Since the functions $x_i(X_a)$ and other quantities may have the jump on $\Omega$ and singularity at points of $\partial\Omega$, in order to transform the integral (2.6) we shall do the following operation. We will replace the region of integration $V_{\Omega}$

by region $V_h$ with the interior boundary $\Omega_h$ , located a small distance h from $\partial\Omega$ (Fig.3). Taking the integral (2.6) over the region $V_h$ by part and then letting h approach zero, we obtain

$$\delta \int_{V_{\Omega^\varepsilon}} \rho_0 f \, dX = \int_{V_\Omega} ( -T_{ia,a}\delta y_i - \mu_{ab,b}\delta Y_a ) \, dX +$$

$$+ \int_{\Omega} \Big[ ( -T^+_{ia}\delta y^+_i + T^-_{ia}\delta y^-_i ) N_a + ( -\mu^+_{ab} + \mu^-_{ab} ) N_b \, \delta Y_a \Big] dX -$$

$$- \int_{\partial\Omega} J_a \delta Y_a \, dS + \int_{\partial V_T} T_{ia} N_a \delta y_i \, dA \qquad (2.7)$$

Here dS is the element of length, the indexes +,- indicate the limit values of quantities on two sides of $\Omega$ and $N_a$ is the outward unit normal vector on the surfaces (on $\Omega$ it is in the direction +). Finally, $J_a$ is the vector of the energy flux entering into the tip of the gap to be calculated by

$$J_a = \lim_{|\Gamma|\Rightarrow 0} \int_{\Gamma} \mu_{ab}\kappa_b \, dS = \lim_{|\Gamma|\Rightarrow 0} \int_{\Gamma} (-T_{ib}x_{i,a}\kappa_b + \rho_0 f \kappa_a ) \, dS \qquad (2.8)$$

where the closed contour $\Gamma$, settling on the transversal to $\Omega$ plane surface, surrounds the point $X_a \in \partial\Omega$ and shrinks to it when the contour length $|\Gamma|$ tends toward zero, and where $\kappa_a$ is the outward unit normal vector on $\Gamma$. The integrals (2.8) resemble the J-integrals in the geometrically linear fracture mechanics [10,11,24]. Note that when deriving (2.8) the following asymptotic formula is assumed to be valid

$$\lim_{|\Gamma|\Rightarrow 0} \int_{\Gamma} T_{ia} \kappa_a \, dS = 0 \qquad (2.9)$$

The property (2.9) means that the generalized force constricting the gap tip vanishes. It takes place if the singularity degree of the stress field $T_{ia}$ is less than $r^{-1}$ where r is the radius from $\Gamma$ to the point $X_a \in \partial\Omega$.

To calculate the second term of the energy functional (2.3), we will suppose at first that $\Omega^\varepsilon$ is smooth. Using a curvilinear, two-dimensional coordinate system $\eta_\alpha$ ($\alpha$=1,2) on the surface $\Omega$, we write

$$\delta \int_{\Omega^{\varepsilon}} 2\gamma \, dA = \delta \int_{\Omega} 2\gamma \sqrt{A_{\varepsilon}} \, d^2\eta = \int_{\Omega} 2\gamma \, \delta\sqrt{A_{\varepsilon}} \, d^2\eta \tag{2.10}$$

Where

$$A_{\varepsilon} = \det|A^{\varepsilon}_{\alpha\beta}| , \quad A^{\varepsilon}_{\alpha\beta} = Y_{a,\alpha} Y_{a,\beta}$$

It is easy to show, that

$$\delta \sqrt{A_{\varepsilon}} = \sqrt{A} \; A^{\alpha\beta} X_{a,\alpha} \delta Y_{a,\beta} \tag{2.11}$$

Here the Greek indices run from 1 to 2, $A=\det|A_{\alpha\beta}|$, $A_{\alpha\beta}=X_{a,\alpha}X_{a,\beta}$ corresponds to the metric tensor of the initial surface $\Omega$, and $A^{\alpha\beta}$ is transverse tensor relative to $A_{\alpha\beta}$. The comma preceeding the Greek index denotes the covariant derivative on the surface $\Omega$. Using (2.11) and taking the integral (2.10) by part we obtain the formula for the variation of surface energy

$$\delta \int_{\Omega^{\varepsilon}} 2\gamma \, dA = \int_{\Omega} 2\gamma \, A^{\alpha\beta} X_{a,\alpha} \delta Y_{a,\beta} \, dA = - \int_{\Omega} 4HN_a \delta Y_a \, dA + \int_{\partial\Omega} 2\gamma \, \nu_a \delta Y_a \, dS \tag{2.12}$$

where $N_a$ is the unit normal vector on $\Omega$, $H=\frac{1}{2} A^{\alpha\beta} X_{a,\alpha\beta} N_a$ is the average curvature of $\Omega$ , and $\nu_a$ is the surface vector, normal with respect to the boundary $\delta\Omega$. One can show that (2.12) remains valid in the case of non-smooth continuation from $\Omega$ to $\Omega^{\varepsilon}$. However, now $\nu_a$ is no longer the tangential vector on $\Omega$ and should be interpreted as the unit normal vector on $\partial\Omega$ denoting the transversal direction of continuation from $\Omega$ to $\Omega^{\varepsilon}$ (Fig.4). The variation of potential of mass force and traction can be calculated in analogical manner

$$\delta \int_{V_{\Omega^{\varepsilon}}} \rho_0 \Phi \, dX = \int_{V_{\Omega}} \rho_0 F_i (-\delta y_i + x_{i,a} \delta Y_a) \, dX - \int_{\Omega} \rho_0 (\Phi^{+} - \Phi^{-}) N_a \delta Y_a \, dA$$

$$\delta \int_{\partial V_T} T_i y_i \, dA = \int_{\partial V_T} T_i \delta y_i \, dA \tag{2.13}$$

where $F_i=-\partial\Phi/\partial x_i$ is the mass force. Combining the formulae (2.7),(2.12),(2.13) one obtains the final expression of the variation of total energy [17-19]

$$\delta\mathcal{E} = \int_{V_{\Omega}} \Big[ -(T_{ia,a} + \rho_0 F_i)\delta y_i + (-\mu_{ab,b} + \rho_0 F_i x_{i,a})\delta Y_a \Big] dX +$$

$$+ \int_{\Omega} \Big\{ (-T^{+}_{ia}\delta y^{+}_i + T^{-}_{ia}\delta y^{-}_i)N_a + \Big[(-\mu^{+}_{ab} + \mu^{-}_{ab})N_b - 4\gamma HN_a - \rho_0(\Phi^{+}-\Phi^{-})N_a\Big]\delta Y_a \Big\} dA +$$

$$+\int_{\partial\Omega} (\, 2\gamma\nu_a - J_a)\delta Y_a dS + \int_{\partial V_T} (T_{ia}N_a - T_i)\delta y_i dA \tag{2.14}$$

We shall now analyze the variational inequality (2.5) taking into account the formula (2.14). It is obvious that $\delta y_i$ and $\delta Y_a$ can be arbitrary in the region $V_\Omega$ as well as $\delta y_i$ on $\partial V_T$. Therefore, by choosing the different signs of $\delta y_i$ and $\delta Y_a$ in a neighbourhood of an arbitrary point of $V_\Omega$ and $\partial V_T$, from (2.5) and (2.14) it follows

$$T_{ia,a} + \rho_0 F_i = 0 \ , \ T_{ia} = \rho_0 \frac{\partial f}{\partial x_{i,a}} \quad \text{in} \quad V_\Omega \tag{2.15}$$

$$-\mu_{ab,b} + \rho_0 F_i x_{i,a} = 0 \ , \ \mu_{ab} = -T_{ib}x_{i,a} + \rho_0 f \,\delta_{ab} \tag{2.16}$$

$$x_i = r_i(X_a) \ \text{on} \ \partial V_x \quad , \quad T_{ia}N_a = T_i \quad \text{on} \ \partial V_T \tag{2.17}$$

Note, however, that the equations (2.16) become identities by virtue of the statical equations (2.15). The cause of that is the invariance of the energy functional relative to the choice of parametrization $Y_a(X_a,\varepsilon)$ inside the region $V_\Omega$.

With the precision of the free energy density $f(x_{i,a})$ the equations (2.15) result in three statical equations concerning three unknown functions $x_i(X_a)$. Usually f depends upon $x_{i,a}$ via the Green's strain tensor $\varepsilon_{ab}$

$$f = f(\varepsilon_{ab}) \ , \quad \varepsilon_{ab} = \frac{1}{2}\,(x_{i,a}x_{i,b} - \delta_{ab})$$

In this case for $T_{ia}$ we have

$$T_{ia} = \sigma_{ab}x_{i,b} \quad , \quad \sigma_{ab} = \rho_0 \frac{\partial f}{\partial \varepsilon_{ab}}$$

The tensor $\sigma_{ab}$ is called the second Piola-Kirchhof stress tensor.

To derive the rest of the relations, let us find the restrictions which must be satisfied by $\delta y_i$ and $\delta Y_a$ on $\Omega$ and $\partial\Omega$. The variation $\delta y_i$ can obviously have arbitrary values on $\Omega$ if the banks of the gap are not in contact with each other in the deformed configuration. If this is not the case we will denote sub-areas of $\Omega$ whose points after deformation will be in contact with each other by $\Omega^+$ and $\Omega^-$

$$x_i^+(\eta_\alpha) = x_i^-(\varphi_\alpha) \ , \ \eta_\alpha \in \Omega^+ \ , \ \varphi_\alpha \in \Omega^- \ , \ \alpha=1,2$$

For those points one can show that

$$[\delta y_i^+(\eta_\alpha) - \delta y_i^-(\theta_\alpha)] n_i \geq 0 \tag{2.18}$$

where $n_i$ is the common unit normal vector on the deformed contact surfaces $\omega^+$ in the direction + (Fig.5). The formula (2.18) can be interpreted as the condition of impenetrability of the gap banks for every admissible configuration. Simultaneously, $\delta y_i^{\mp} x_{i,\alpha}$ can have arbitrary values on $\Omega^{\mp}$, where $x_{i,\alpha} = \partial x_i / \partial \eta_\alpha$ . On supposing $\Omega^\varepsilon \supseteq \Omega$ we have the following restrictions for $\delta Y_a$

$$\delta Y_a N_a = 0 \quad \text{on } \Omega$$

$$\delta Y_a \nu_a \geq 0 \ , \quad \delta Y_a \pi_a = 0 \ , \quad \delta Y_a \tau_a \ \text{- arbitrary on } \partial\Omega \tag{2.19}$$

where $\tau_a$ is the tangential vector on $\partial\Omega$ and where the vectors $\tau_a, \nu_a, \pi_a$ form the orthonormal base at the point $X_a \in \partial\Omega$.

Taking into account all above restrictions, one can easily show that (2.5), (2.14) and (2.15)-(2.19) lead to

$$T_{ia}^{\mp} N_a = 0 \text{ on } \Omega \setminus \Omega^{\mp} \tag{2.20}$$

$$T_{ia}^{+} N_a \sqrt{A/a} \ \Big|_{\eta_\alpha} = T_{ia}^{-} N_a \sqrt{A/a} \ \Big|_{\theta_\alpha} = -p n_i \ , \ p \geq 0 \quad \text{on } \Omega^+ \tag{2.21}$$

$$(-\mu_{ab}^{+} + \mu_{ab}^{-}) \ N_b X_{a,\alpha} = 0 \quad \text{on } \Omega \tag{2.22}$$

$$|J_\alpha| = \sqrt{J_a J_a} \leq 2\gamma \ , \qquad J_3 = J_a \tau_a = 0 \quad \text{on } \partial\Omega \tag{2.23}$$

where

$$A = \det|A_{\alpha\beta}| \ , \qquad A_{\alpha\beta} = X_{a,\alpha} X_{a,\beta}$$

$$a = \det|a_{\alpha\beta}| \ , \qquad a_{\alpha\beta} = x_{a,\alpha} x_{a,\beta}$$

The tensors $A_{\alpha\beta}$ and $a_{\alpha\beta}$ correspond to the metrics of the initial and deformed surfaces of the gap respectively. If we use the Cauchy's stress tensor $t_{ij}$ instead of $T_{ia}$ and the relation [52-53]

$$t_{ij}n_j\sqrt{a} = T_{ia}N_a\sqrt{A}$$

the condition (2.21) takes the simple form

$$t^+_{ij}n_j = t^-_{ij}n_j = -p'n_i \ , \qquad p' \geq 0 \quad \text{on } \omega^+$$

It is easy to see that the condition (2.22) becomes an identity by virtue of other conditions. In fact, using the definition of $\mu_{ab}$ and the conditions (2.20), (2.21) we have

$$\mu^{\mp}_{ab}N_bX_{a,\alpha} = -T^{\mp}_{ib}x_{i,a}N_bX_{a,\alpha} + \rho_0 f^{\mp}N_aX_{a,\alpha} = -T^{\mp}_{ib}N_bx_{i,\alpha} = 0$$

The cause of the last identity is again the invariance of the energy functional relative to the choice of parametrization $Y_a(X_a,\varepsilon)$, which transforms the surface $\Omega$ into itself. We shall prove that the condition $J_3=0$ is also the consequence of the other conditions of (2.15)-(2.23). Let us consider the equilibrium configuration as a comparison configuration relative to itself. Then $\delta\mathcal{E}=0$ by definition of the variation. We then construct a family of parametrizations $Y_a(X_a,\varepsilon)$ transforming V and $\Omega$ into themselves. From the formulae (2.14)-(2.23), which hold for the equilibrium configuration, one has

$$\delta\mathcal{E} = -\int_{\partial\Omega} J_3 \ \delta Y_a \ \tau_a \ dS = 0$$

Therefore, due to the arbitrariness of $\delta Y_a\tau_a$ on $\partial\Omega$, this relations leads to $J_3\equiv 0$.

Thus, the relations (2.15), (2.17), (2.20), (2.21), (2.23) compose the system of statical equations and boundary conditions which should be satisfied by any equilibrium configuration. It is of interest to note that the condition (2.23) is separated from the rest of the relations. Therefore, in practice one can first solve the system (2.15), (2.17), (2.20), (2.21) to find the deformed configuration $x_i(X_a)$ and the stress field $T_{ia}$. Then, by using (2.9), we calculate the J-integrals and verify the inequality (2.23). It can be easily shown that the system (2.15), (2.17), (2.20), (2.21) follows from the variational inequality

$$\delta\mathscr{E} \geq 0 \tag{2.24}$$

which holds for arbitrary family of admissible configurations with the fixed surface of discontinuity $\Omega$. To prove this we need only the choice $Y_a(X_a,\varepsilon)\equiv X_a$ and the formula (2.14) with $\delta Y_a\equiv 0$ in $V_\Omega$, on $\Omega$ and on $\partial\Omega$. From the inequality (2.24) one can directly go to discretization for the numerical approach of concrete problems [54].

## 3. Quasistatics of a geometrically nonlinear elastic-plastic body with a crack

As it was noted in the Introduction, the stress singularity described by the elasticity theory cannot be realized in the neighbourhood of the gap tip because of the yield condition. Therefore, for a more adequate description of the material behaviour in this zone one should use the model of an elastic-plastic body. Our aim is to generalize the variational principle of total energy to quasistatics of an elastic-plastic body with a singular surface propagating throughout the body. Such a propagating surface of material discontinuity will be called a crack.

Let us first study the mathematical model of a geometrically nonlinear elastic-plastic body without crack and the variational formulation of the quasistatical boundary value problem in the plasticity theory [26-27]. Let the initial configuration of the body occupying the region V of the Euclidean three-dimensional space be chosen for a reference configuration. We denote the Cartesian co-ordinates of a typical point of this configuration by $X_a$, $a=1,2,3$. At a time $\tau$, this material point has the Cartesian co-ordinates $x_i$ given by

$$x_i = x_i(X_1, X_2, X_3, \tau) , \quad X_a \in V \tag{3.1}$$

The co-ordinates $x_i$ run over the region $v_\tau$ of the deformed configurations. At an arbitrary instant of time the functions $x_i(X_a, \tau)$ perform a one-to-one continuously- differentiable transformation of V into $v_\tau$, which is called a deformation. A one-parameter family of deformation $x_i(X_a, \tau)$ perform a motion of the body. We shall suppose that the kind of motion to be considered in this chapter is quasistatical in the sense that the velocity and acceleration of particles of the body are negligibly small compared with other quantities. The general case of dynamics will be consider in the next chapter. Such a quasistatical motion of an elastic-plastic body is charaterized along with the functions $x_i(X_a,\tau)$ by an additional measure of plastic strain $\varepsilon^p_{ab}(X_a,\tau)$ and hardening parameters $\chi_A(X_a,\tau)$, A=1,...,N (see [25-27]). Note immediately that the plastic strain tensor $\varepsilon^p_{ab}$ does not in general satisfy the equation of compatibility. The functions $\varepsilon^p_{ab}(X_a,\tau)$ and $\chi_A(X_a,\tau)$ are referred to as internal degrees of freedom [3]. With the introduced internal degrees of freedom the free energy per unit mass of an elastic-plastic body is given by the formula

$$f = f(\varepsilon_{ab}-\varepsilon^p_{ab},\chi_A,\vartheta) \tag{3.2}$$

$$\varepsilon_{ab} = \frac{1}{2}\,(x_{i,a}x_{i,b} - \delta_{ab})$$

Here $\varepsilon_{ab}$ is the symmetric Green's strain tensor, $\chi_A$ correspond to the hardening parameters, while $\vartheta$ denotes the fixed temperature. The temperature is assumed to remain constant, therefore, it is not necessary to list it among the variables of the function f. The model of an elastic-plastic body will be established by constructing the closed system of equations concerning $x_i(X_a,\tau)$, $\varepsilon^p_{ab}(X_a,\tau)$, and $\chi_A(X\ ,\tau)$. For this purpose let us introduce the following notations

$$e^p = (\varepsilon^p_{ab},\chi_A)$$

$$s = \left( \rho_0\frac{\partial f}{\partial\varepsilon_{ab}}\ ,\ -\rho_0\frac{\partial f}{\partial\chi_A} \right) \tag{3.3}$$

All of these objects have 6+N components, among which the first six are of a symmetric tensor, while the rest are of a generalized vector. The generalized tensor s depends upon the variables $\varepsilon_{ab}$, $\varepsilon^p_{ab}$, $\chi_A$. The motive of these notations will become clear in the next paragraph.

We will now postulate the yield condition and the generalized associated law [27-32] for the internal degrees of freedom $e^p$

$$\mathcal{F}(s) \le 0 \tag{3.4}$$

$$(\hat{s} - s) : \dot{e}^p \le 0\ ,\quad \forall\ \hat{s}\ ,\ \mathcal{F}(\hat{s}) \le 0 \tag{3.5}$$

Here $\mathcal{F}$ is the convex function depending on 6+N variables of the generalized tensor $\mathbf{s}$ , $\mathcal{F}:\mathbb{R}^{6+N}\Rightarrow\mathbb{R}$ (see Fig.6), $\hat{\mathbf{s}}$ is the following generalized tensor : $\hat{\mathbf{s}}=(\hat{\sigma}_{ab},\hat{\pi}_A)$. The two dots in the right-hand expression of the inequality (3.5) are used to denote the inner product of two generalized tensors

$$(\hat{\mathbf{s}} - \mathbf{s}) : \dot{\mathbf{e}}^P = (\hat{\sigma}_{ab} - \rho_0\frac{\partial f}{\partial\varepsilon_{ab}})\,\dot{\varepsilon}^P_{ab} + (\hat{\pi}_A + \rho_0\frac{\partial f}{\partial\chi_A})\,\dot{\chi}_A$$

The quantities $\dot{\varepsilon}^P_{ab}$ and $\dot{\chi}_A$ correspond to the plastic strain rate and the rate of hardening parameters

$$\dot{\varepsilon}^P_{ab} = \frac{\partial}{\partial\tau}\,\varepsilon^P_{ab}(X_a,\tau)\ ,\quad \dot{\chi}_A = \frac{\partial}{\partial\tau}\,\chi_A(X_a,\tau)$$

It should be noted that all the restrictions (3.4), (3.5) are referred to as the category of non-holonomical constraints [3]. Therefore, the plasticity theory has evident irreversibility.

For the determination of $x_i(X_a,\tau)$, let us consider at an arbitrary fixed time t the set $\mathcal{E}_t$ of all admissible configurations $y_i(X_a,t)$ of the body satisfying the kinematical condition

$$y_i(X_a,t) = r_i(X_a)\ ,\quad X_a\in\partial V_x\subset\partial V$$

On this set of admissible configurations we define the energy functional of the elastic-plastic body by the expression

$$\mathcal{E}[y_i(X_a)] = \int_V \rho_0 f(\varepsilon_{ab}-\varepsilon^P_{ab},\chi_A)dX + \int_V \rho_0\Phi(x_i)\ dX - \int_{\partial V_T} T_i x_i dA \qquad (3.6)$$

In the energy functional (3.6) $\partial V=\partial V_T\cup\partial V_x$ , $\varepsilon_{ab} = \frac{1}{2}\,(y_{i,a}y_{i,b}-\delta_{ab})$ denotes the Green's strain tensor, $\varepsilon^P_{ab}(X_a,t)$ and $\chi_A(X_a,t)$ should be considered as the fixed functions. The other symbols, such as $\rho_0$, $\Phi(y_i)$, $T_i$ have the same sense as before (cf. the functional (2.3)). In comparison with the elasticity theory there are two modifications: a) the internal degrees of freedom $\varepsilon^P_{ab}$ and $\chi_A$ are contained in the free energy density as the given function, b) there is an implicit time dependence of the energy functional. For simplicity, we shall suppose that the potential of mass force $\Phi(y_i)$, the position radius-vector

$r_i(X_a)$ and the "dead" traction $T_i(X_a)$ do not depend explicitly on time.

The variation of the energy functional at the actual configuration $x_i(X_a,t)$ with respect to an arbitrary family of admissible configurations $y_i(X_a,t,\varepsilon)\in \mathscr{C}_t$ will be defined as follows

$$\delta_x \mathscr{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathscr{E}[y_i(X_a,t,\varepsilon)] \tag{3.7}$$

Here , as before, a one-parameter family of admissible configurations $y_i(X_a,t,\varepsilon)\in \mathscr{C}_t$ is supposed to be continuously differentiable, satisfying at the same time the condition

$$y_i(X_a,t,0) = x_i(X_a,t)$$

The index x in $\delta_x\mathscr{E}$ indicates that only the external degrees of freedom in the energy functional are subject to variation. In contrast, the internal degrees of freedom should be considered as the fixed functions.

Now we will postulate the following variational principle: at the actual configuration of an elastic-plastic body the variation of the energy functional vanishes for arbitrary family of admissible comparison configurations at arbitrary time t

$$\delta_x\mathscr{E} = 0 \quad , \quad \forall\ y_i(X_a,t,\varepsilon)\in \mathscr{C}_t \tag{3.8}$$

It is easy to show that (3.8) is equivalent to

$$\delta_x\mathscr{E} = -\int_V (T_{ia,a} + \rho_0 F_i)\ \delta y_i\ dX + \int_{\partial V_T} (T_{ia}N_a - T_i)\ \delta y_i\ dA = 0 \tag{3.9}$$

where

$$T_{ia} = \rho_0 \frac{\partial f}{\partial \varepsilon_{ab}} x_{i,b} \quad , \quad F_i = -\frac{\partial \Phi}{\partial x_i} \tag{3.10}$$

$$\delta y_i = \frac{\partial}{\partial \varepsilon}\Big|_{X_a,t=\mathrm{const},\varepsilon=0} y_i(X_a,t,\varepsilon)$$

By choosing arbitrary values of $\delta y_i$ in V and on $\partial V_T$ , from the variational equation (3.9) one derives the quasistatical system of equations and boundary conditions

$$T_{ia,a} + \rho_0 F_i = 0 \quad \text{in } V \tag{3.11}$$

$$x_i = r_i(X_a) \text{ on } \partial V_x \quad , \quad T_{ia}N_a = T_i \text{ on } \partial V_T$$

With the precision of the free energy $f(\varepsilon_{ab}-\varepsilon^p_{ab},\chi_A)$ and the yield function $\mathcal{F}(s)$, the relations (3.4), (3.5), (3.10), (3.11) become closed for the determination of 9+N unknown functions $x_i(X_a,\tau)$, $\varepsilon^p_{ab}(X_a,\tau)$, $\chi_A(X_a,\tau)$. In the case of small elastic strain [27], the free energy density is approximated by

$$\rho_0 f = \frac{1}{2}\ [\ C_{abcd}(\varepsilon_{ab}-\varepsilon^p_{ab})(\varepsilon_{cd}-\varepsilon^p_{cd}) + Z_{AB}\chi_A\chi_B\ ] \tag{3.12}$$

We might also suppose that there is an interaction between the strain and the hardening effect so that the following formula is valid

$$\rho_0 f = \frac{1}{2}\ [\ C_{abcd}(\varepsilon_{ab}-\varepsilon^p_{ab})(\varepsilon_{cd}-\varepsilon^p_{cd}) + 2H_{abA}(\varepsilon_{ab}-\varepsilon^p_{ab})\chi_A + Z_{AB}\chi_A\chi_B\ ] \tag{3.13}$$

With the acception of (3.12) we rewrite all the basic equations of the plasticity theory in the form

$$T_{ia,a} + \rho_0 F_i = 0 \quad , \quad T_{ia} = \sigma_{ab}x_{i,b} \tag{3.14}$$

$$\varepsilon_{ab} = \frac{1}{2}\ (x_{i,a}x_{i,b}-\delta_{ab}) \tag{3.15}$$

$$\sigma_{ab} = C_{abcd}(\varepsilon_{cd}-\varepsilon^p_{cd}) \quad , \quad \pi_A = -Z_{AB}\chi_B \tag{3.16}$$

$$\mathcal{F}(\sigma_{ab},\pi_A) \le 0 \tag{3.17}$$

$$(\hat{\sigma}_{ab}-\sigma_{ab})\dot{\varepsilon}^p_{ab} + (\hat{\pi}_A-\pi_A)\dot{\chi}_A \le 0 \quad , \quad \forall\ (\hat{\sigma}_{ab},\hat{\pi}_A),\ \ \mathcal{F}(\hat{\sigma}_{ab},\hat{\pi}_A) \le 0 \tag{3.18}$$

$$x_i = r_i(X_a) \text{ on } \partial V_x \quad , \quad T_{ia}N_a = T_i \text{ on } \partial V_T \tag{3.19}$$

The sense of the variational equation (3.8),(3.9) becomes clear if we choose the real motion of the body for comparison configurations

$$y_i(X_a,t,\varepsilon) = x_i(X_a,\tau) = x_i(X_a,t+\varepsilon)$$

In this case the variation $\delta y_i$ in (3.9) will be replaced by $\dot{x}_i$ , where

$$\dot{x}_i = \frac{\partial x_i}{\partial \varepsilon}\Big|_{X_a=\text{const},\,\varepsilon=0} = \frac{\partial x_i}{\partial \varepsilon}\Big|_{X_a=\text{const},\,\tau=t}$$

Therefore, from (3.9) we obtain the equality

$$-\int_V (T_{ia,a} + \rho_0 F_i)\, \dot{x}_i \, dX + \int_{\partial V_T} (T_{ia}N_a - T_i)\, \dot{x}_i \, dA = 0 \tag{3.20}$$

Using the Gauss-Ostrogradski's theorem, we transform (3.20) to

$$\int_V \sigma_{ab}\dot{\varepsilon}_{ab} \, dX - \int_V \rho_0 F_i \dot{x}_i \, dX - \int_{\partial V_T} T_i \dot{x}_i \, dA = 0 \tag{3.21}$$

Substracting and adding $\sigma_{ab}\dot{\varepsilon}^{P}_{ab}$ and $\pi_A\dot{\chi}_A$ , we can rewrite (3.21) in the form

$$\frac{d}{d\tau}\Big|_{\tau=t} \int_V \rho_0 f \, dX - \int_V \rho_0 F_i \dot{x}_i \, dX - \int_{\partial V_T} T_i \dot{x}_i \, dA + \int_V (\sigma_{ab}\dot{\varepsilon}^{P}_{ab} + \pi_A\dot{\chi}_A) \, dX = 0 \tag{3.22}$$

The equality (3.22) is evidently the equation of energy balance for the whole region of the body at the time $\tau=t$. The terms $\sigma_{ab}\dot{\varepsilon}^{P}_{ab} + \pi_A\dot{\chi}_A$ correspond to the non-negative dissipation of energy [27].

Consider now the elastic-plastic body whose initial configuration contains a gap. This gap is supposed to settle on the smooth surface $\Omega_0$ with the boundary $\partial\Omega_0$. The initial configuration occupies the region $V_0 = V\setminus(\Omega_0 \cup \partial\Omega_0)$. The co-ordinates of a typical point are denoted by $X_a$, $X_a \in V_0$. At some time $\tau$, due to the slow propagation of the crack and the deformation, the co-ordinates of this particle point are given by

$$x_i = x_i(X_a, \tau) \; , \qquad X_a \in V_\tau = V\setminus(\Omega_\tau \cup \partial\Omega_\tau) \tag{3.23}$$

where $\Omega_\tau$ is the surface of material discontinuity at the time $\tau$, satisfying the restriction

$$\Omega_\tau \supseteq \Omega_t \quad , \quad \tau > t$$

The "material" configuration of the body occupying the region $V_\tau$ will be chosen for a reference configuration at the time $\tau$. Thus, the peculiarity of the quasistatical crack problem is that even in a material description one has a changable interior boundary, which is not specified *a priori*. Our aim is to construct a closed system of basic equations concerning the law of motion $x_i(X_a,\tau)$, the internal degrees of freedom $\varepsilon^{P}_{ab}(X_a,\tau)$, $\chi_A(X_a,\tau)$ and also the unknown interior boundary $\Omega_\tau U\partial\Omega_\tau$.

For this purpose, we introduce a set $\mathscr{C}_t$ of all admissible configurations $y_i(X_a,t)$, compared with the actual configuration $x_i(X_a,t)$ at some arbitrary time t. Every admissible configuration should have a surface of discontinuity $\Sigma_t$ containing $\Omega_t$. As before, the smoothness of $\Sigma_t$ is assumed, except at points of $\Omega_t$, where $\Sigma_t$ may have non-smooth continuation of $\Omega_t$. The singularity of the deformation gradients $\partial y_i/\partial X_a$ at $\partial\Sigma_t$ is accepted and the kinematical condition

$$y_i(X_a,t) = r_i(X_a) \ , \quad X_a \in \partial V_x \subset \partial V$$

is supposed to be satisfied.

On the arbitrary admissible comparison configuration $y_i(X_a,t)\in\mathscr{C}_t$, the total energy functional of an elastic-plastic cracked body can be defined as follows

$$\mathscr{E}[y_i(X_a,t)]=\int_{V_{\Sigma_t}} \rho_0 f(\varepsilon_{ab}(y_{i,a})-\varepsilon^P_{ab},\chi_A)dX+\int_{\Sigma_t} 2\gamma dA+\int_{V_{\Sigma_t}} \rho_0\Phi(y_i)dX-\int_{V_T} T_i y_i dA \qquad (3.24)$$

where

$$V_{\Sigma_t} = V\backslash(\Sigma_t U\partial\Sigma_t) \quad , \quad \varepsilon_{ab}(y_{i,a}) = \frac{1}{2} \ (y_{i,a}y_{i,b}-\delta_{ab})$$

The tensor $\varepsilon^P_{ab}(X_a,t)$ corresponds to the symmetric tensor of plastic strain, while $\chi_A(X_a,t)$ are the hardening parameters. The quantities $\varepsilon^P_{ab}$ and $\chi_A$ should be considered as the given functions in the total energy functional. The other symbols, such as $\rho_0$, f, $\gamma$, $\Phi$, $T_i$ are explained above.

To define a variation of the functional (3.24), consider a family of admissible comparison configurations $y_i(X_a,t,\varepsilon)\in\mathscr{C}_t$ , with surfaces of discontinuity $\Omega^\varepsilon_t$ . As before, this family should satisfy the following conditions

$$\Omega^{\varepsilon'}_t \geq \Omega^\varepsilon_t \geq \Omega_t \quad , \ \varepsilon'>\varepsilon>0 \quad ; \quad \Omega^\varepsilon_t \Rightarrow \Omega_t \quad \text{when } \varepsilon\Rightarrow 0$$

$$y_i(X_a,t,0) = x_i(X_a,t)$$

The variation of the total energy functional at the actual configuration with respect to an arbitrary family of admissible comparison configurations will be defined as follows

$$\delta_x \mathcal{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}[y_i(X_a,t,\varepsilon)] \tag{3.25}$$

Now we postulate the following inequalities which hold for the real quasistatical motion of an elastic-plastic cracked body

$$\mathcal{F}(s) \leq 0 \tag{3.26}$$

$$(\hat{s} - s) : \dot{e}^p \leq 0 \ , \quad \forall\, \hat{s}\ , \ \mathcal{F}(\hat{s}) \leq 0 \tag{3.27}$$

$$\delta_x \mathcal{E} \geq 0 \quad , \quad \forall\ y_i(X_a,t,\varepsilon) \in \mathcal{E}_t \ , \forall t \tag{3.28}$$

where the symbols $e^p$, $s$ and $\hat{s}$ denote the generalized tensors defined by (3.3). Futhermore, we shall demand the exact equality from (3.28) as an equation of energy balance, if the real motion $x_i(X_a,\tau) = x_i(X_a,t+\varepsilon)$ is considered as the family of admissible configurations in comparison with the actual configuration $x_i(X_a,t)$.

As in the second chapter, it is convenient to introduce a family of parametrizations of medium $Y_a(X_a,t)$ for the calculation of $\delta_x\mathcal{E}$, so that

$$V_t \overset{Y_a}{\Rightarrow} V_t^{\varepsilon} = V\setminus(\Omega_t^{\varepsilon} \cup \partial\Omega_t^{\varepsilon}) \quad , \quad \Omega_t \overset{Y_a}{\Rightarrow} \Omega_t^{\varepsilon}$$

$$Y_a(X_a,t,\varepsilon) = X_a \quad \text{when } \varepsilon=0 \text{ or } X_a \in \partial V$$

By these parametrizations, one can show that

$$\delta_x \mathcal{E} = \int_{V_t} \Big[-(T_{ia,a}+\rho_0 F_i)\delta y_i + (-\mu_{ab,b} - \sigma_{bc}\varepsilon^p_{bc,a} - \pi_A \chi_{A,a} + \rho_0 F_i x_{i,a})\delta Y_a\Big] dX +$$

$$+ \int_{\Omega_t} \Big\{(-T^+_{ia}\delta y^+_i + T^-_{ia}\delta y^-_i)N_a + [(-\mu^+_{ab}+\mu^-_{ab})N_b - 4\gamma H N_a - \rho_0(\Phi^+ - \Phi^-)]\delta Y_a\Big\}\, dA +$$

$$+\int_{\partial\Omega_t}(2\gamma\nu_a - I_a)\delta Y_a dS + \int_{\partial V_t}(T_{ia}N_a - T_i)\delta y_i dA \geq 0 \qquad (3.29)$$

Here the symbol $\delta$ under the integral sign denotes the derivative with respect to $\varepsilon$ when $X_a$, t are held constant

$$\delta y_i = \frac{\partial}{\partial\varepsilon}\Big|_{X_a,t=\mathrm{const},\varepsilon=0} y_i(Y_a(X_a,t,\varepsilon),t,\varepsilon)$$

$$\delta Y_a = \frac{\partial}{\partial\varepsilon}\Big|_{X_a,t=\mathrm{const},\varepsilon=0} Y_a(X_a,t,\varepsilon)$$

The tensor $T_{ai}$, $\sigma_{ab}$, $\pi_A$ and $\mu_{ab}$ are given by the formulae

$$T_{ia} = \sigma_{ab}x_{i,b} \quad , \quad \sigma_{ab} = \rho_0\frac{\partial f}{\partial\varepsilon_{ab}}$$

$$\pi_A = -\rho_0\frac{\partial f}{\partial\chi_A} \quad , \quad \mu_{ab} = -T_{ib}x_{i,a} + \rho_0 f\delta_{ab}$$

The flux of potential energy entering into the crack tip is calculated by the J-integrals

$$J_a = \lim_{|\Gamma_t|\Rightarrow 0}\int_{\Gamma_t}(-T_{ib}x_{i,a}\kappa_b + \rho_0 f\kappa_a)dS$$

where the contour $\Gamma_t$ , settling on the transversal to $\partial\Omega_t$ plane surface, surrounds the point $X_a\in\partial\Omega_t$ and shrinks to it when the contour length $|\Gamma_t|$ tends towards zero, and where $\kappa_a$ is the outward unit normal vector on $\Gamma_t$. The other symbols in (3.29) have the same meaning as before (cf. the formulae (2.7)-(2.14)). It is easy to show that $\delta y_i$ and $\delta Y_a$ should satisfy the following constraints

$$[\delta y_i^+(\eta_\alpha) - \delta y_i^-(\theta_\alpha)]n_i \geq 0 \quad , \quad \eta_\alpha\in\Omega_t^+,\ \varphi_\alpha\in\Omega_t^-$$

$$\delta Y_a N_a = 0 \quad \text{on } \Omega_t \qquad (3.31)$$

$$\delta Y_a\nu_a \geq 0\ ,\quad \delta Y_a\pi_a = 0\ ,\quad \delta Y_a\tau_a \text{ - arbitrary on } \partial\Omega_t$$

Here $\Omega_t^+$ and $\Omega_t^-$ denote sub-areas of $\Omega_t$ whose points at time t are in contact with each other: $x_i^+(\eta_\alpha) = x_i^-(\varphi_\alpha)$ , $\eta_\alpha\in\Omega^+$ , $\varphi_\alpha\in\Omega^-$ , $\alpha$=1,2; $n_i$ is the unit normal vector on the common deformed contact surfaces $\omega^+$ . The vectors $\tau_a$, $\nu_a$,

$\pi_a$ compose the orthonormal base at the point $X_a \in \partial\Omega_t$, where $\tau_a$ is the tangential vector of $\partial\Omega_t$.

The inequality (3.29) with the constraints (3.31) lead to

$$T_{ia,a} + \rho_0 F_i = 0 \quad , \quad T_{ia} = \sigma_{ab} x_{i,b} \quad , \quad \sigma_{ab} = \rho_0 \frac{\partial f}{\partial \varepsilon_{ab}} \quad \text{in } V_t$$

$$x_i = r_i(X_a) \text{ on } \partial V_x \quad , \quad T_{ia} N_a = T_i \quad \text{on } \partial V_T$$

$$T^{\mp}_{ia} N_a = 0 \text{ on } \Omega_t \setminus \Omega^{\mp}_t \tag{3.32}$$

$$T^{+}_{ia} N_a \sqrt{A/a}\ \Big|_{\eta_\alpha} = T^{-}_{ia} N_a \sqrt{A/a}\ \Big|_{\theta_\alpha} = -p n_i \ , \ p \geq 0 \quad \text{on } \Omega^{+}_t$$

$$|J_\alpha| = \sqrt{J_a J_a} \leq 2\gamma \ , \quad J_3 = J_a \tau_a = 0 \quad \text{on } \partial\Omega_t$$

Now let the real motion of the cracked body $x_i(X_a,\tau) = x_i(X_a, t+\varepsilon)$, $X_a \in V_t$ be taken for the family of admissible configurations. Accordingly, the family of parametrizations $Y_a(X_a, t, \varepsilon)$ will be replaced by the functions $X'_a(X_a, t+\varepsilon)$ mapping $\Omega_t$ into $\Omega_{t+\varepsilon}$ and $V_t$ into $V_{t+\varepsilon}$ and satisfying the condition

$$X'_a(X_a, t+\varepsilon) = X_a \qquad \text{when } \varepsilon = 0 \text{ or } X_a \in \partial V$$

With the changes

$$\delta y_i \ \Rightarrow \ \delta_t x_i = \frac{\partial}{\partial \tau}\Big|_{X_a = \text{const}, \tau = t} x_i(X'_a(X_a,\tau),\tau)$$

$$\delta Y_a \ \Rightarrow \ \dot{X}'_a = \frac{\partial}{\partial \tau}\Big|_{X_a = \text{const}, \tau = t} X'_a(X_a,\tau)$$

the inequality (3.29) should turn into the exact equality expressing the equation of energy balance. With the account of (3.32) from (3.29) we obtain

$$\int_{\Omega_t} (-T^{+}_{ia} \delta_t x^{+}_i + T^{-}_{ia} \delta_t x^{-}_i) N_a dA + \int_{\partial\Omega_t} (2\gamma\lambda_a - J_a) \dot{X}'_a dS = 0 \tag{3.33}$$

where $\lambda_a$ denotes the normal direction of crack propagation from $\Omega_t$ into $\Omega_{t+\varepsilon}$. The equation (3.33) leads to the following additional conditions

$$\left.\begin{array}{l} p > 0 \ \Rightarrow \ [\dot{x}^{+}_i(\eta_\alpha) - \dot{x}^{-}_i(\theta_\alpha)] n_i = 0 \\ p = 0 \ \Rightarrow \ [\dot{x}^{+}_i(\eta_\alpha) - \dot{x}^{-}_i(\theta_\alpha)] n_i \geq 0 \end{array}\right\} \text{on } \Omega^{+}_t \tag{3.34}$$

$$\left.\begin{array}{l} |J_\alpha| < 2\gamma \;\Rightarrow\; \dot{X}'_a = 0 \quad \text{(no propagation)} \\ |J_\alpha| = 2\gamma \;\Rightarrow\; \dot{X}'_a\lambda_a \geq 0\ ,\quad 2\gamma\lambda_a = J_a \end{array}\right\} \text{ on } \partial\Omega_t$$

where the velocity of particles $\dot{x}_i$ is given by

$$\dot{x}_i = \frac{\partial}{\partial\tau}\Big|_{X_a=\text{const},\tau=t} x_i(X_a,\tau)$$

Note that the last condition of (3.34) defines the direction of crack propagation by way of $J_a$. Thus, the crack will propagate along the direction with the maximum module of the flux of energy which is equal to $2\gamma$.

The relations (3.26), (3.27), (3.32), (3.34) compose the closed system of equations and boundary conditions, which must be used to find $x_i(X_a,\tau)$, $\varepsilon^p_{ab}(X_a,\tau)$, $\chi_A(X_a,\tau)$ and the surface $\Omega_t$ . We can also consider the limit case of an elastic-perfectly-plastic cracked body as a body with the hardening coefficient $Z_{AB}$ tending toward zero. Naturally we expect the following limit relations

$$\mathcal{F}(\sigma_{ab},0) \leq 0$$

$$(\hat{\sigma}_{ab} - \sigma_{ab})\,\dot{\varepsilon}^p_{ab} \leq 0\ ,\quad \forall\,\hat{\sigma}_{ab},\ \mathcal{F}(\hat{\sigma}_{ab},0) \leq 0$$

$$T_{ia,a} + \rho_0 F_i = 0\ ,\ T_{ia} = \sigma_{ab}x_{i,b}\ ,\quad \sigma_{ab} = \rho_0\frac{\partial f}{\partial\varepsilon_{ab}} \quad \text{in } V_t$$

$$x_i = r_i(X_a) \text{ on } \partial V_x\ ,\quad T_{ia}N_a = T_i \text{ on } \partial V_T$$

$$T^{\mp}_{ia}N_a = 0 \text{ on } \Omega_t\backslash\Omega^{\mp}_t \tag{3.35}$$

$$T^{+}_{ia}N_a\sqrt{A/a}\,\Big|_{\eta_\alpha} = T^{-}_{ia}N_a\sqrt{A/a}\,\Big|_{\theta_\alpha} = -pn_i\ ,\ p\geq 0 \text{ on } \Omega^+_t$$

$$|J_\alpha| = \sqrt{J_aJ_a} \leq 2\gamma\ ,\quad J_3 = J_a\tau_a = 0 \text{ on } \partial\Omega_t$$

$$\left.\begin{array}{l} p > 0 \;\Rightarrow\; [\dot{x}^+_i(\eta_\alpha) - \dot{x}^-_i(\theta_\alpha)]n_i = 0 \\ p = 0 \;\Rightarrow\; [\dot{x}^+_i(\eta_\alpha) - \dot{x}^-_i(\theta_\alpha)]n_i \geq 0 \end{array}\right\} \text{ on } \Omega^+_t$$

$$\left.\begin{array}{l} |J_\alpha| < 2\gamma \;\Rightarrow\; \dot{X}'_a = 0 \quad \text{(no propagation)} \\ |J_\alpha| = 2\gamma \;\Rightarrow\; \dot{X}'_a\lambda_a \geq 0\ ,\quad 2\gamma\lambda_a = J_a \end{array}\right\} \text{ on } \partial\Omega_t$$

By $J_a$ we mean the following limit J-integrals

$$J_a = \lim_{Z_{AB}\Rightarrow 0} \lim_{|\Gamma_t|\Rightarrow 0} \int_{\Gamma_t} (-T^Z_{ib} x^Z_{i,a} \kappa_b + \rho_0 f^Z \kappa_a) \, dS \tag{3.36}$$

where the index Z indicates the solution of the elastic-plastic crack problem with the small hardening coefficients $Z_{AB}$. Note that we cannot write the following limit formula for $J_a$

$$J_a = \lim_{|\Gamma_t|\Rightarrow 0} \int_{\Gamma_t} (-T_{ib} x_{i,a} \kappa_b + \rho_0 f \kappa_a) dS$$

because of the fact, that the deformation gradients cannot be uniquely defined in the limit problem (3.35).

## 4. Dynamical theory of fracture.

Consider now the situation, when the equilibrium or quasistatical motion of a cracked body is not possible, and the crack will quickly propagate throughout the body until its separation into pieces. Obviously, the account of dynamical factors is then necessary. It will be shown in this chapter that this account can be made within the framework of a variational inequality of evolution [17,19,46-48]. We shall study first a motion of an elastic body with a propagating crack. Such the motion is described by the following law

$$x_i = x_i(X_a,\tau)\ , \qquad X_a \in V_\tau = V\backslash(\Omega_\tau \cup \partial\Omega_\tau) \tag{4.1}$$

where $\Omega_\tau$ is the surface of discontinuity at the time $\tau$. At arbitrary time the functions $x_i(X_a,\tau)$ compose a one-to-one continuously differentiable transformation of $V_\tau$ into some open region, say $v_\tau$. Note that the material configuration $V_\tau$ , which is chosen to be a reference configuration at the time $\tau$, depends upon the time. Therefore, even in the material description the dynamical crack problem is the problem with the unknown interior boundary $\Omega_\tau$ , which has to be found along with other quantities, such as the stress, the deformation gradients, etc. As before, we allow the jump of $x_i(X_a,\tau)$ on $\Omega_\tau$ and the singularity of the deformation gradients $\partial x_i/\partial X_a$ at points of $\partial\Omega_\tau$ . We allow also the singularity of the particle velocity $\dot{x}_i$ at points of $\partial\Omega_\tau$ , where the dot over a function denotes its time derivative, when $X_a$ are held constant

$$\dot{x}_i = \frac{\partial}{\partial\tau}\Big|_{X_a=\mathrm{const}} x_i(X_a,\tau)$$

We shall analyze now the equation of energy balance for an elastic body with a crack, which obviously has the form

$$\frac{d}{d\tau}\left( \mathcal{E} + \mathcal{T} \right) = 0 \tag{4.2}$$

where

$$\mathcal{E} = \int_{V_\tau} \rho_0 u(\varepsilon_{ab}, s)\, dX + \int_{\Omega_\tau} 2\gamma dA + \int_{V_\tau} \rho_0 \Phi(x_i) dX - \int_{\partial V_T} T_i x_i dA \tag{4.3}$$

$$\mathcal{T} = \int_{V_\tau} \frac{1}{2} \rho_0 \dot{x}_i \dot{x}_i dX \tag{4.4}$$

The functional $\mathcal{E}$ corresponds to the total potential energy of the body plus the surface energy of the crack, while the functional $\mathcal{T}$ describes the kinetic energy of the body. Here $u(\varepsilon_{ab}, s)$ is the internal energy of the body per unit mass, $\varepsilon_{ab}=1/2(x_{i,a}x_{i,b}-\delta_{ab})$ is the Green's strain tensor, s denotes the entropy, which is supposed to be constant within the process of motion under consideration. Because the entropy remains constant, it is not necessary to list it among the variables of the function u. The other symbols, such as $\rho_0$, $\gamma$, $\Phi(x_i)$, $T_i$, $\dot{x}_i$ are explained above. Because of the changable regions $V_\tau$ and surfaces $\Omega_\tau$ of integration, in order to calculate $\dot{\mathcal{E}}$ and $\dot{\mathcal{T}}$ at the time $\tau=t$ it is convenient to introduce a family of parametrization $X'_a=X'_a(X_a,\tau)$ so that

$$V_t \overset{X'_a}{\Rightarrow} V_\tau \quad , \quad \Omega_t \overset{X'_a}{\Rightarrow} \Omega_\tau$$

$$X'_a(X_a,\tau) = X_a \quad \text{when } \tau=t \text{ or } X_a\in\partial V$$

Note, that the functional $\mathcal{E}$ of (4.3) depends upon $\tau$ in the same way as $\mathcal{E}$ of (2.3) upon $\varepsilon$. Therefore, it is easy to see that

$$\begin{aligned}\dot{\mathcal{E}} = &\int_{V_t} \left[-(T_{ia,a}+\rho_0 F_i)\delta_t x_i + (-\mu_{ab,b}+\rho_0 F_i x_{i,a})\dot{X}'_a\right] dX + \\ &+ \int_{\Omega_t} \left[(-T^+_{ia}\delta_t x^+_i + T^-_{ia}\delta_t x^-_i)N_a + (-\mu^+_{ab}+\mu^-_{ab})N_b \dot{X}'_a\right] dA + \\ &+ \int_{\partial\Omega_t} (2\gamma\nu_a - J_a)\dot{X}'_a dS + \int_{\partial V_T} (T_{ia}N_a - T_i)\delta_t x_i dA\end{aligned} \tag{4.5}$$

When writing (4.5) the property $\dot{X}'_a N_a=0$ on $\Omega_t$ was taken into account. In the equation (4.5) the stress tensor $T_{ia}$ and the Eshelby's tensor $\mu_{ab}$ are given by

$$T_{ia} = \sigma_{ab} x_{i,b} \quad , \quad \sigma_{ab} = \rho_0 \frac{\partial u}{\partial \varepsilon_{ab}}$$

$$\mu_{ab} = -T_{ib}x_{i,a} + \rho_0 u\delta_{ab}$$

The flux of potential energy entering into the crack tip is calculated by the contour integrals

$$J_a = \lim_{|\Gamma_t|\Rightarrow 0} \int_{\Gamma_t} (-T_{ib}x_{i,a}\kappa_b + \rho_0 u\kappa_a)dS$$

where the contour $\Gamma_t$ , settling on the transversal to $\partial\Omega_t$ plane surface, surrounds the point $X_a \in \partial\Omega_t$ and shrinks to it when the contour length $|\Gamma_t|$ tends towards zero. The symbol $\delta_t x_i$ denotes partial derivative of the composite function with respect to $\tau$ where $X_a$ are held constant

$$\delta_t x = \frac{\partial}{\partial\tau} \Big|_{X_a=\text{const},\tau=t} x_i(X'_a(X_a,\tau),\tau)$$

It should be emphasized that $\delta_t x_i$ and the particle velocity $\dot{x}_i$ are distinguished from each other. Let occupy ourself now with the calculation of $\dot{\mathcal{T}}$

$$\dot{\mathcal{T}} = \frac{d}{d\tau} \Big|_{\tau=t} \int_{V_t} \frac{1}{2} \rho_0 \dot{x}_i \dot{x}_i \det\left|\frac{\partial X'_a}{\partial X_b}\right| dX =$$

$$= \int_{V_t} \delta_t \left( \frac{1}{2} \rho_0 \dot{x}_i \dot{x}_i \det\left|\frac{\partial X'_a}{\partial X_b}\right| \right) dX =$$

$$= \int_{V_t} \left( \rho_0 \dot{x}_i \delta_t \dot{x}_i + \frac{1}{2} \rho_0 \dot{x}_i \dot{x}_i \dot{X}'_{a,a} \right) dX \qquad (4.6)$$

It is easy to see that

$$\delta_t x_i = \dot{x}_i + x_{i,a}\dot{X}'_a \quad , \quad \delta_t \dot{x}_i = \ddot{x}_i + \dot{x}_{i,a}\dot{X}'_a \qquad (4.7)$$

where $\ddot{x}_i$ is the acceleration of particles. Substituting $\delta_t \dot{x}_t$ from (4.7) into (4.6) and using the Gauss-Ostrogradski's theorem one obtains

$$\dot{\mathcal{T}} = \int_{V_t} \rho_0 \ddot{x}_i (\delta_t x_i - x_{i,a}\dot{X}'_a) \, dX - \int_{\partial\Omega_t} Q_a \dot{X}'_a \, dS \qquad (4.8)$$

where

$$Q_a = \lim_{|\Gamma_t|\Rightarrow 0} \int_{\Gamma_t} -\frac{1}{2} \rho_0 \dot{x}_i \dot{x}_i \kappa_a dS \qquad (4.9)$$

The vector $Q_a$ can be interpreted as the flux of kinetic energy entering into the tip of the crack. If the velocity field has no singularity at points of $\partial\Omega_t$, then $Q_a$ vanishes. But in general case we shall suppose that $Q_a$ is non-trivial.

Thus, from (4.5)-(4.9) one obtains the equation of energy balance for the cracked body in the form

$$\frac{d}{d\tau}\Big|_{\tau=t}(\mathcal{E}+\mathcal{T})=\int_{V_t}\left[(\rho_0\dot{x}_i-T_{ia,a}-\rho_0F_i)\delta_t x_i+(-\mu_{ab,b}+\rho_0F_ix_{i,a}-\rho_0\dot{x}_ix_{i,a})\dot{X}'_a\right]dX+$$
$$+\int_{\Omega_t}\left[(-T^+_{ia}\delta_t x^+_i+T^-_{ia}\delta_t x^-_i)N_a+(-\mu^+_{ab}+\mu^-_{ab})N_b\dot{X}'_a\right]dA+$$
$$+\int_{\partial\Omega_t}(2\gamma\nu_a-I_a)\dot{X}'_a dS+\int_{\partial V_t}(T_{ia}N_a-T_i)\delta_t x_i dA=0 \qquad (4.10)$$

where the vector $I_a$ denotes the total flux of energy entering into the crack tip to be calculated by

$$I_a=\lim_{|\Gamma_t|\Rightarrow 0}\int_{\Gamma_t}\left[-T_{ib}x_{i,a}\kappa_b+\rho_0(u+\frac{1}{2}\dot{x}_i\dot{x}_i)\kappa_a\right]dS \qquad (4.11)$$

Let us begin the consideration of the variational inequality of evolution of fracture mechanics. For this purpose, we shall introduce two sets. The first set is the set of admissible comparison configurations $\mathcal{E}_t$, which consists of all one-to-one continuously piecewise differentiable transformations having singular surfaces $\Sigma_t\supseteq\Omega_t$ and satisfying the kinematical condition

$$y_i(X_a,t)=r_i(X_a)\quad,\quad X_a\in\partial V_x$$

The second set denoted by $\mathcal{P}_t$ is the set of all parametrizations $Y_a(X_a,t)$ satisfying the following conditions

$$V_t\overset{Y_a}{\Rightarrow}V_{\Sigma_t}=V\setminus(\Sigma_t\cup\partial\Sigma_t)\quad,\quad\Omega_t\overset{Y_a}{\Rightarrow}\Sigma_t$$

$$Y_a(X_a,t)=X_a\quad\text{when } X_a\in\partial V$$

On the set $\mathcal{E}_t$ we define the energy functional as follows

$$\mathcal{E}[y_i(X_a,t)] = \int_{V_{\Sigma_t}} \rho_0 u(\varepsilon_{ab}(y_{i,a}))\, dX + \int_{\Sigma_t} 2\gamma dA + \int_{V_{\Sigma_t}} \rho_0 \Phi(y_i) dX - \int_{\partial V_T} T_i y_i dA$$

$$\varepsilon_{ab}(y_{i,a}) = \frac{1}{2}\,(y_{i,a} y_{i,b} - \delta_{ab})$$

Now we postulate the following variational principle: at the actual configuration of the elastic cracked body the variational inequality of evolution

$$\delta\mathcal{E} + \int_{V_t} \rho_0 \ddot{x}_i (\delta y_i - x_{i,a}\delta Y_a)\, dX - \int_{\partial\Omega_t} Q_a \delta Y_a\, dS \geq 0 \tag{4.12}$$

holds at arbitrary time t for arbitrary variations of comparison configurations $\delta y_i$ and parametrizations $\delta Y_a$. Moreover, this variational principle demands the absolute equality from (4.12) as an energy balance equation, if $\delta y_i$ and $\delta Y_a$ are replaced by $\delta_t x_i$ and $\dot{X}'_a$. The variation in the inequality (4.12) should be defined as follows

$$\delta\mathcal{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}[y_i(X_a,t,\varepsilon)]$$

$$\delta y_i = \frac{\partial}{\partial\varepsilon}\Big|_{X_a,t=\mathrm{const},\varepsilon=0} y_i(Y_a(X_a,t,\varepsilon),t,\varepsilon)$$

$$\delta Y_a = \frac{\partial}{\partial\varepsilon}\Big|_{X_a,t=\mathrm{const},\varepsilon=0} Y_a(X_a,t,\varepsilon)$$

Here $y_i(X_a,t,\varepsilon)\in \mathcal{E}_t$ and $Y_a(X_a,t,\varepsilon)\in \mathcal{P}_t$ are arbitrary generalized curves in the corresponding sets ,satisfying the conditions

$$y_i(X_a,t,0) = x_i(X_a,t)$$

$$Y_a(X_a,t,0) = X_a$$

In the inequality (4.12) the second term correspond to the virtual work of the usual d'Alembert's force of inertia. It should be noted that this term does not depend upon the parametrizations $Y_a(X_a,t,\varepsilon)$. In fact, from the definition of variations we have

$$\delta y_i - x_{i,a}\delta Y_a = \frac{\partial}{\partial\varepsilon}\Big|_{X_a,t=\mathrm{const},\varepsilon=0} y_i(X_a,t,\varepsilon)$$

The last term of (4.12) is non-classical. The cause of its appearance can be explained by comparing (4.12) with (4.8),(4.10). However, the analysis of the energy balance equation is only the heuristic argument for the acceptance of the postulate (4.12).

Analogically with the variational calculation done before, from (4.12) we have

$$\int_{V_t} \left[(\rho_0 \ddot{x}_i - T_{ia,a} - \rho_0 F_i)\delta y_i + (-\mu_{ab,b} + \rho_0 F_i x_{i,a} - \rho_0 \ddot{x}_i x_{i,a})\delta Y_a\right] dX +$$

$$+ \int_{\Omega_t} \left[(-T^+_{ia}\delta y^+_i + T^-_{ia}\delta y^-_i)N_a + (-\mu^+_{ab} + \mu^-_{ab})N_b \delta Y_a\right] dA +$$

$$+ \int_{\partial\Omega_t} (2\gamma\nu_a - I_a)\delta Y_a dS + \int_{\partial V_t} (T_{ia}N_a - T_i)\delta y_i dA \geq 0 \tag{4.13}$$

The inequality (4.13) with the kinematical constraints on $\delta y_i$ and $\delta Y_a$ (cf. the formula (3.31)) leads to the relations

$$T_{ia,a} + \rho_0 F_i = \rho_0 \ddot{x}_i \ , \quad T_{ia} = \rho_0 \frac{\partial u}{\partial \varepsilon_{ab}} x_{i,b} \ \text{in } V_t$$

$$x_i = r_i(X_a) \ \text{on } \partial V_x \ , \quad T_{ia}N_a = T_i \ \text{on } \partial V_T$$

$$T^{\mp}_{ia}N_a = 0 \ \text{on } \Omega_t \backslash \Omega^{\mp}_t \tag{4.14}$$

$$T^+_{ia}N_a\sqrt{A/a}\,|_{\eta_\alpha} = T^-_{ia}N_a\sqrt{A/a}\,|_{\varphi_\alpha} = -pn_i, \ p \geq 0 \quad \text{on } \Omega^+_t$$

$$|I_\alpha| = \sqrt{I_a I_a} \leq 2\gamma \ , \quad I_3 = I_a\tau_a = 0 \quad \text{on } \partial\Omega_t$$

The last condition of (4.14) on $\partial\Omega_t$ has the following physical sense: the module of the transversal total flux of energy entering into the crack tip should be less than, or equal to, the doubled surface energy density. Obviously, this condition is not enough for the definition of the direction of crack propagation. To obtain more information we use the second part of the formulated postulate. By replacing $\delta y_i$ and $\delta Y_a$ by $\delta_t x_i$ and $\dot{X}'_a$ in (4.13) and taking (4.14) into account, we reduce (4.13) to

$$\int_{\Omega_t} (-T^+_{ia}\delta_t x^+_i + T^-_{ia}\delta_t x^-_i)N_a dA + \int_{\partial\Omega_t} (2\gamma\lambda_a - I_a)\dot{X}'_a dS = 0 \tag{4.15}$$

Here the vector $\lambda_a$ indicates the transversal direction of crack propagation, and, correspondingly, $\lambda_a \dot{X}'_a$ is the transversal velocity of the crack front. Thus, the equality (4.15) leads to the following relationship

$$\begin{array}{ll} p > 0 \;\Rightarrow\; [\dot{x}_i^+(\eta_\alpha) - \dot{x}_i^-(\theta_\alpha)]n_i = 0 & \text{on } \Omega_t^+ \\ p = 0 \;\Rightarrow\; [\dot{x}_i^+(\eta_\alpha) - \dot{x}_i^-(\theta_\alpha)]n_i \ge 0 & \\ |I_\alpha| < 2\gamma \;\Rightarrow\; \dot{X}'_a = 0 \quad \text{(no propagation)} & \text{on } \partial\Omega_t \\ |I_\alpha| = 2\gamma \;\Rightarrow\; \dot{X}'_a \lambda_a \ge 0 \,, \quad 2\gamma\lambda_a = I_a & \end{array} \tag{4.16}$$

The last condition of (4.16) determines the direction of crack propagation that coincides with the direction of the vector $I_a$. The crack will therefore propagate along the direction with the maximum module of energy flux, which is equal to the doubled surface energy density.

The relations (4.14),(4.16) together compose the closed boundary value crack problems, which must be used to find the motion of the cracked body as well as the contact surfaces and the crack front.

All of this construction can easily be generalized to fracture dynamics of an elastic-plastic cracked body. Let the actual configuration of such the body be described by

$$x_i = x_i(X_a, t) \,, \qquad X_a \in V_t = V \setminus (\Omega_t \cup \partial\Omega_t) \tag{4.17}$$

The goal is to find these functions and also the internal degrees of freedom $\varepsilon^P_{ab}(X_a,t)$ and $\chi_A(X_a,t)$ corresponding to the symmetric plastic strain tensor and the hardening parameters respectively. For this purpose we shall now introduce the set of admissible comparison configurations $\mathcal{E}_t$ and parametrizations $\mathcal{P}_t$ as it has been done before. On the set $\mathcal{E}_t$ we define the energy functional of an elastic-plastic body as follows

$$\mathcal{E}[y_i(X_a,t)] = \int_{V_{\Sigma_t}} \rho_0 u(\varepsilon_{ab}(y_{i,a}) - \varepsilon^P_{ab}, \chi_A)dX + \int_{\Sigma_t} 2\gamma dA + \int_{V_{\Sigma_t}} \rho_0 \Phi(y_i)dX - \int_{V_T} T_i y_i dA \tag{4.18}$$

where $\varepsilon^P_{ab}(X_a,t)$ and $\chi_A(X_a,t)$ must be considered as the fixed functions. The theory is characterized also by the following convex function

$$\mathcal{F} : \mathbb{R}^{6+N} \Rightarrow \mathbb{R}$$

which is called the yield function.

Consider arbitrary one-parameter families of comparison configurations $y_i(X_a, t, \varepsilon) \in \mathcal{C}_t$ and parametrizations $Y_a(X_a, t, \varepsilon) \in \mathcal{P}_t$ with the conditions

$$y_i(X_a, t, 0) = x_i(X_a, t)$$

$$Y_a(X_a, t, 0) = X_a$$

Then the principle of dynamics of an elastic-plastic cracked body states : at the real process of motion of an elastic-plastic cracked body the following inequalities

$$\mathcal{F}\left( \rho_0 \frac{\partial u}{\partial \varepsilon_{ab}} , -\rho \frac{\partial u}{\partial \chi_A} \right) \le 0 \tag{4.19}$$

$$\left( \hat{\sigma}_{ab} - \rho_0 \frac{\partial u}{\partial \varepsilon_{ab}} \right) \dot{\varepsilon}_{ab} + \left( \hat{\pi}_A + \rho_0 \frac{\partial u}{\partial \chi_A} \right) \dot{\chi}_A \le 0 \quad \forall\, \hat{\sigma}_{ab} , \hat{\pi}_A \tag{4.20}$$

$$\mathcal{F}(\hat{\sigma}_{ab}, \hat{\pi}_A) \le 0 \tag{4.21}$$

$$\delta_x \mathcal{E} + \int_{V_t} \rho_0 \ddot{x}_i (\delta y_i - x_{i,a} \delta Y_a)\, dX - \int_{\partial\Omega_t} Q_a \delta Y_a \, dS \ge 0 \tag{4.22}$$

take place at arbitrary time and for all $y_i \in \mathcal{C}_t$ and $Y_a \in \mathcal{P}_t$. Here the variation of energy functional is defined by the following formula

$$\delta \mathcal{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}[y_i(X_a, t, \varepsilon)] \tag{4.23}$$

We demand also the absolute equality from (4.22) as the energy balance equation, if $\delta y_i$ and $\delta Y_a$ will be replaced by $\delta_t x_i$ and $\dot{X}'_a$.

From (4.19)-(4.22) we obtain the following system of basic equations for the elastic-plastic cracked body

$$T_{ia,a} + \rho_0 F_i = \rho_0 \ddot{x}_i \ , \quad T_{ia} = \sigma_{ab} x_{i,b} \ \text{in } V_t$$

$$\sigma_{ab} = \rho_0 \frac{\partial u}{\partial \varepsilon_{ab}} \ , \quad \pi_A = -\rho_0 \frac{\partial u}{\partial \chi_A}$$

$$\mathcal{F}(\sigma_{ab}, \pi_A) \le 0$$

$$(\hat{\sigma}_{ab} - \sigma_{ab})\, \dot{\varepsilon}^{p}_{ab} + (\hat{\pi}_A - \pi_A)\, \dot{\chi}_A \le 0 \quad , \quad \forall\, (\hat{\sigma}_{ab}, \hat{\pi}_A)$$

$$\mathcal{F}(\hat{\sigma}_{ab}, \hat{\pi}_A) \le 0$$

$$x_i = r_i(X_a) \text{ on } \partial V_x \quad , \quad T_{ia}N_a = T_i \text{ on } \partial V_T \tag{4.24}$$

$$T^{\mp}_{ia}N_a = 0 \text{ on } \Omega_t \backslash \Omega^{\mp}_t$$

$$T^{+}_{ia}N_a\sqrt{A/a}\,\big|_{\eta_\alpha} = T^{-}_{ia}N_a\sqrt{A/a}\,\big|_{\varphi_\alpha} = -pn_i, \quad p \ge 0 \quad \text{on } \Omega^{+}_t$$

$$|I_\alpha| = \sqrt{I_aI_a} \le 2\gamma \;, \qquad I_3 = I_a\tau_a = 0 \qquad \text{on } \partial\Omega_t$$

$$\left.\begin{array}{l} p > 0 \;\Rightarrow\; [\dot{x}^{+}_i(\eta_\alpha) - \dot{x}^{-}_i(\theta_\alpha)]n_i = 0 \\ p = 0 \;\Rightarrow\; [\dot{x}^{+}_i(\eta_\alpha) - \dot{x}^{-}_i(\theta_\alpha)]n_i \ge 0 \end{array}\right. \quad \text{on } \Omega^{+}_t$$

$$\left.\begin{array}{l} |I_\alpha| < 2\gamma \;\Rightarrow\; \dot{X}'_a = 0 \quad \text{(no propagation)} \\ |I_\alpha| = 2\gamma \;\Rightarrow\; \dot{X}'_a\lambda_a \ge 0 \;,\quad 2\gamma\lambda_a = I_a \end{array}\right. \quad \text{on } \partial\Omega_t$$

Here the I-integrals are defined by the formula (4.11) as in the elastodynamical crack problem.

From (4.24) one can also obtain the limit problem of an elastic-perfectly-plastic cracked body by letting the hardening coefficient approach zero. We do not write the limit relations and I-integrals in view of the analogical formulae (3.35)-(3.36).

## 5. Geometrically linear theory of fracture.

In this chapter the geometrically linear variants of the constructed fracture mechanics are presented.

1. Geometrically linear statics of an elastic gapped body.

Let us consider a gapped body occupying the region $V_\Omega = V \setminus (\Omega \cup \partial\Omega)$ in the natural configuration. We shall suppose that under the influence of external forces and tractions material points of the body will have a small displacement $w_a(X_a)$ given by

$$w_a(X_a) = x_a(X_a) - X_a \quad , \quad X_a \in V_\Omega \tag{5.1}$$

The goal is to construct a system of basic equations for the equilibrium dispacement field. According to the principle of total energy, at the equilibrium displacement field the variation of the total energy functional of an elastic gapped body should be non-negative for an arbitrary family of admissible comparison displacements. We will now define the set of admissible comparison displacements, the total energy functional of an elastic gapped body and its variation.

Let us introduce the Sobolev vector space $(H^1_\Sigma)^3$ consisting of all square integrable displacement fields with square integrable derivatives on $V_\Sigma$ and the kinematical constraints on $\partial V_x$

$$(H^1_\Sigma)^3 = \{ \ v_a \mid v_a \text{ and } v_{a,b} \in L^1_\Sigma \ , \ v_a|_{\partial V_x} = 0 \ \}$$

The norm of a displacement field $v_a$ on this space can be defined as follows

$$\|v\|^2 = \int_{V_\Sigma} (v_a v_a + v_{a,b} v_{a,b}) \ dX$$

Note that due to the non-smoothness of the region $V_\Sigma$ the condition of square integrability of $v_a$ and $v_{a,b}$ on $V_\Sigma$ permits the singularity of $v_{a,b}$ at points of $\partial\Sigma$.

Let us consider the following set $\mathscr{C}_\Sigma \subset (H^1_\Sigma)^3$

$$\mathcal{C}_\Sigma = \{ \, v_a \mid v_a \in (H^1_\Sigma)^3 \; , \; (v^+_a - v^-_a)N_a \geq 0 \; , \; X_a \in \Sigma \, \} \tag{5.2}$$

Here the indices + and - indicate the limit values (traces) of quantities on two sides of the singular surface, $N_a$ is the unit normal vector taken outer to a side + of $\Sigma$. It is easy to see that $\mathcal{C}_\Sigma$ is convex. Now the set $\mathcal{C}$ of admissible comparison displacements can be defined as the union of $\mathcal{C}_\Sigma$ with $\Sigma \supseteq \Omega$

$$\mathcal{C} = \bigcup_{\Sigma \supseteq \Omega} \mathcal{C}_\Sigma \tag{5.3}$$

For an arbitrary admissible displacement field $v_a(X_a) \in \mathcal{C}$ with a singular surface $\Sigma$ we define the total energy functional of an elastic gapped body as follows

$$\mathcal{E}[v_a(X_a)] = \int_{V_\Sigma} \rho_0 f(\varepsilon_{ab}(v), \vartheta) dX + \int_\Sigma 2\gamma \, dA + \int_{V_\Sigma} \rho_0 F_a v_a \, dX - \int_{\partial V_T} T_a v_a dA \tag{5.4}$$

Here the strain tensor $\varepsilon_{ab}$ has the following linear form

$$\varepsilon_{ab}(v_a) = \frac{1}{2} \left( v_{a,b} + v_{b,a} \right) \tag{5.5}$$

The other symbols, such as $\rho_0$, $f$, $\vartheta$, $\gamma$, $F_a$, $T_a$ have the same meaning as before.

Consider a one-parameter family of admissible comparison displacements $v_a(X_a,\varepsilon) \in \mathcal{C}$, with the singular surfaces $\Omega^\varepsilon$, satisfying the following conditions

$$\Omega^{\varepsilon'} \supseteq \Omega^\varepsilon \supseteq \Omega \quad \text{for } \varepsilon' > \varepsilon > 0, \qquad \Omega^\varepsilon \Rightarrow \Omega \quad \text{when } \varepsilon \Rightarrow 0$$

$$v_a(X_a, 0) = w_a(X_a)$$

The variation of the total energy functional at the actual displacement field with relation to the family $v_a(X_a,\varepsilon) \in \mathcal{C}$ is defined as follows

$$\delta\mathcal{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}[v_a(X_a,\varepsilon)] \tag{5.6}$$

The variational principle of total energy states

$$\delta\mathcal{E} \geq 0 \; , \qquad \forall \; v_a(X_a,\varepsilon) \in \mathcal{C} \tag{5.7}$$

if $w_a(X_a)$ is the equilibrium displacement field.

For the calculation of $\delta\mathcal{E}$, we construct a family of parametrizations $Y_a(X_a,\varepsilon)\in\mathcal{P}$ transforming $V_\Omega$ into $V_{\Omega^\varepsilon}$ , $\Omega$ into $\Omega^\varepsilon$ and satisfying the conditions

$$Y_a(X_a,\varepsilon) = X_a \quad \text{when } \varepsilon=0 \text{ or } X_a\in \partial V$$

By this family of parametrizations one can show that

$$\delta\mathcal{E} = \int_{V_\Omega} \Big[ -(\sigma_{ab,b}+ \rho_0 F_a)\delta v_a + (-\mu_{ab,b} + \rho_0 F_b w_{b,a})\delta Y_a\Big] dX +$$

$$+ \int_\Omega \Big\{ (-\sigma^+_{ab}\delta v^+_a + \sigma^-_{ab}\delta v^-_a)N_b + \Big[(-\mu^+_{ab}+ \mu^-_{ab})N_b-4\gamma HN_a-\rho_0(F^+_b w^+_b-F^-_b w^-_b)N_a\Big]\delta Y_a \Big\}dA +$$

$$+ \int_{\partial\Omega} ( 2\gamma\nu_a- J_a)\delta Y_a dS + \int_{\partial V_T} (\sigma_{ab}N_b - T_a)\delta v_a dA \tag{5.8}$$

Here $\sigma_{ab}$ is the symmetric stress tensor defined by

$$\sigma_{ab} = \rho_0 \frac{\partial f}{\partial\varepsilon_{ab}} \tag{5.9}$$

The Eshelby's tensor $\mu_{ab}$ has the form

$$\mu_{ab} = -\sigma_{cb}w_{c,a} + \rho_0 f \delta_{ab} \tag{5.10}$$

Accordinly, the J-integrals should be calculated by the formula

$$J_a = \lim_{|\Gamma|\Rightarrow 0} \int_\Gamma \mu_{ab}\kappa_b dS = \lim_{|\Gamma|\Rightarrow 0} \int_\Gamma (-\sigma_{cb}w_{c,a}\kappa_b + \rho_0 f\kappa_a ) dS \tag{5.11}$$

The other symbols in (5.8) can be interpreted as before.

From (5.7),(5.8) and (5.2) follows

$$\sigma_{ab,b}+ \rho_0 F_a = 0 , \quad \sigma_{ab} = \rho_0 \frac{\partial f}{\partial\varepsilon_{ab}}$$

$$\varepsilon_{ab} = \frac{1}{2} (w_{a,b} + w_{b,a})$$

$$w_a= 0 \quad \text{on } \partial V_x \quad , \quad \sigma_{ab}N_b = T_a \quad \text{on } \partial V_T$$

$$(w^+_a - w^-_a)N_a \geq 0 \quad \text{on } \Omega \tag{5.12}$$

$$\sigma^+_{ab}N_b = \sigma^-_{ab}N_b = -pN_a , \quad p\geq 0 \quad \text{on } \Omega$$

$$(w^+_a - w^-_a)N_a > 0 \Rightarrow p = 0$$

$$|J_\alpha| = \sqrt{J_a J_a} \le 2\gamma \ , \qquad J_3 = J_a \tau_a = 0 \quad \text{on } \partial\Omega$$

If the free energy density is approximated by the quadratic form

$$\rho_0 f = \frac{1}{2} C_{abcd} \varepsilon_{ab} \varepsilon_{cd}$$

then (5.12) is reduced to the classical crack problem [10] with the linear constitutive relation

$$\sigma_{ab} = C_{abcd} \varepsilon_{cd} \tag{5.13}$$

where $C_{abcd}$ is the fourth-rank tensor of elastic constants. We shall admit that they satisfy the symmetry and positivity properties

$$C_{abcd} = C_{bacd} = C_{abdc} = C_{cdab}$$

$$C_{abcd} \varepsilon_{ab} \varepsilon_{cd} \ge \alpha \, \varepsilon_{ab} \varepsilon_{ab} \quad , \quad \alpha > 0 \quad , \quad \forall \, \varepsilon_{ab} \ \text{(symmetric)}$$

It should be emphasized that even in the case of the linear relationship (5.13) the problem (5.12) as a whole remains non-linear. The cause of that is the boundary conditions on $\Omega$ and $\partial\Omega$ expressed by inequalities. This fact plays a very important role in the application of the method of homogenization [48].

Note also, that the last boundary condition of (5.12) is separated from other relations. Therefore in practice one can first solve the problem (5.12) without this condition and then, by calculating the J-integrals, verify its validity. Without the last condition the problem (5.12) can be formulated as the following extremal problem:

Find $w \in \mathscr{E}_\Sigma$ such that

$$\underline{\mathscr{E}} = \mathscr{E}\,[w] = \inf_{v \in \mathscr{E}_\Omega} \mathscr{E}\,[v] \tag{5.14}$$

One can also rewrite the problem (5.14) with the acception of (5.13) in terms of variational inequality:

Find $w \in \mathscr{E}_\Sigma$ such that

$$a(w, v-w) \ge \int_{V_\Sigma} \rho_0 F_a (v_a - w_a)\, dX - \int_{\partial V_T} T_a (v_a - w_a)\, dA \tag{5.15}$$

where the bilinear form $a(\mathbf{w},\mathbf{v})$ on $\mathcal{E}_\Omega$ is given by

$$a(\mathbf{w},\mathbf{v}) = \int_{V_\Sigma} C_{abcd}(w_{a,b} + w_{b,a})(v_{c,d} + v_{d,c})\, dX \tag{5.16}$$

Using the Korn's inequality [46] which holds for the region $V_\Omega$ , one can prove the existence and uniqueness theorem for the variational problem (5.14). One can also apply the finite element method to (5.14) to go to the discrete formulation of the crack problem.

2. Geometrically linear quasistatics of an elastic-plastic cracked body.

The law of quasistatical motion of the cracked body is given by

$$x_a = X_a + w_a(X_a,\tau) \quad , \quad X_a \in V_\tau = V \setminus (\Omega_\tau \cup \partial\Omega_\tau)$$

The goal is to construct a system of basic equations concerning the displacement field $w_a(X_a,\tau)$ , the internal degrees of freedom $\varepsilon^P_{ab}(X_a,\tau)$ and $\chi_A(X_a,\tau)$, and also the unknown interior boundary $\Omega_\tau \cup \partial\Omega_\tau$. We suppose that $\varepsilon^P_{ab}$ is symmetric and $\varepsilon^P_{ab}$ and $\chi_A$ are square-integrable functions on $V_\tau$:

$$\varepsilon^P_{ab}(X_a,\tau) \in (L^2_\tau)^6 \quad , \quad \|\varepsilon^P\|^2 = \int_{V_\tau} \varepsilon^P_{ab}\varepsilon^P_{ab}\, dX$$

$$\chi_A(X_a,\tau) \in (L^2_\tau)^N \quad , \quad \|\chi\|^2 = \int_{V_\tau} \chi_A\chi_A\, dX$$

Let us define the total energy functional of an elastic-plastic cracked body on an arbitrary admissible comparison displacement field $v_a(X_a,\tau) \in \mathcal{E}_\tau$ with a singular surface $\Sigma_\tau$ as follows

$$\mathcal{E}[v_a(X_a,\tau)] = \int_{V_{\Sigma_\tau}} \rho_0 f(\varepsilon_{ab}(v) - \varepsilon^P_{ab}, \chi_A)dX + \int_{\Sigma_\tau} 2\gamma dA + \int_{V_{\Sigma_\tau}} \rho_0 F_a v_a dX - \int_{V_T} T_a v_a dA \tag{5.17}$$

where

$$\varepsilon_{ab}(v) = \frac{1}{2}(v_{a,b} + v_{b,a}) \quad , \quad v_a \in \mathcal{E}_\tau$$

In the functional (5.17) the quantities $\varepsilon^P_{ab}$ and $\chi_A$ should be considered as the fixed functions. The set $\mathcal{E}_\tau$ of all admissible comparison displacement fields $v_a(X_a,\tau)$ at the time $\tau$ is defined as follows

$$\mathscr{E}_\tau = \bigcup_{\Sigma_\tau \supseteq \Omega_\tau} \mathscr{E}_{\Sigma_\tau}$$

The other symbols, such as $\rho_0$, f, $\gamma$, $F_a$, $T_a$ are explained above.

Now we formulate the quasistatical cracked problem for an elastic-plastic cracked body in the following way

Find $w_a \in \mathscr{E}_t$ , $\varepsilon^P_{ab} \in (L^2_t)^6$ , $\chi_A \in (L^2_t)^N$ such that

$$\mathscr{F}\left( \rho_0 \frac{\partial f}{\partial \varepsilon_{ab}} \, , \, -\rho_0 \frac{\partial f}{\partial \chi_A} \right) \le 0 \tag{5.18}$$

$$(\hat{\sigma}_{ab} - \rho_0 \frac{\partial f}{\partial \varepsilon_{ab}}) \, \dot{\varepsilon}^P_{ab} + (\hat{\pi}_A + \rho_0 \frac{\partial f}{\partial \chi_A}) \, \dot{\chi}_A \le 0 \quad , \quad \forall \, (\hat{\sigma}_{ab}, \hat{\pi}_A) \tag{5.19}$$

$$\mathscr{F}(\hat{\sigma}_{ab}, \hat{\pi}_A) \le 0 \tag{5.20}$$

$$\delta_x \mathscr{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathscr{E}[v_a(X_a, t, \varepsilon)] \ge 0 \quad , \quad \forall \, v_a(X_a, t, \varepsilon) \in \mathscr{E}_t \tag{5.21}$$

Moreover, the last inequality (5.21) should turn into the absolute equality expressing the equation of energy balance, if the real displacement field $w_a(X_a, \tau) = w_a(X_a, t+\varepsilon)$ is chosen for the family of admissible comparison displacements. Here $\mathscr{F}: \mathbb{R}^{6+N} \Rightarrow \mathbb{R}$ is the convex yield function, the family of admissible comparison displacements $v_a(X_a, t, \varepsilon) \in \mathscr{E}_t$ should satisfy the condition

$$v_a(X_a, t, 0) = w_a(X_a, t)$$

From the above formulation of the crack problem one can obtain the following relations

$$\mathscr{F}(\sigma_{ab}, \pi_A) \le 0$$

$$(\hat{\sigma}_{ab} - \sigma_{ab})\dot{\varepsilon}^P_{ab} + (\hat{\pi}_A - \pi_A)\dot{\chi}_A \le 0 \quad , \quad \forall \, (\hat{\sigma}_{ab}, \hat{\pi}_A), \quad \mathscr{F}(\hat{\sigma}_{ab}, \hat{\pi}_A) \le 0$$

$$\sigma_{ab,b} + \rho_0 F_a = 0$$

$$\sigma_{ab} = \rho_0 \frac{\partial f}{\partial \varepsilon_{ab}} \quad , \quad \pi_A = -\rho_0 \frac{\partial f}{\partial \chi_A}$$

$$\varepsilon_{ab} = \frac{1}{2} (w_{a,b} + w_{b,a})$$

$$w_a = 0 \quad \text{on } \partial V_x \quad , \quad \sigma_{ab} N_b = T_a \quad \text{on } \partial V_T$$

$$(w_a^+ - w_a^-) N_a \geq 0 \quad \text{on } \Omega_t \tag{5.22}$$

$$\sigma^+_{ab} N_b = \sigma^-_{ab} N_b = -pN_a \; , \; p \geq 0 \quad \text{on } \Omega_t$$

$$(w_a^+ - w_a^-) N_a > 0 \Rightarrow p = 0$$

$$|J_\alpha| = \sqrt{J_a J_a} \leq 2\gamma \; , \quad J_3 = J_a \tau_a = 0 \quad \text{on } \partial\Omega_t$$

$$\left.\begin{array}{l} p > 0 \Rightarrow [\dot{w}_a^+ - \dot{w}_a^-] \, N_a = 0 \\ p = 0 \Rightarrow [\dot{w}_a^+ - \dot{w}_a^-] \, N_a \geq 0 \end{array}\right. \quad \text{on } \Omega_t$$

$$\left.\begin{array}{l} |J_\alpha| < 2\gamma \Rightarrow \dot{X}'_a = 0 \quad \text{(no propagation)} \\ |J_\alpha| = 2\gamma \Rightarrow \dot{X}'_a \lambda_a \geq 0 \; , \quad 2\gamma\lambda_a = J_a \end{array}\right. \quad \text{on } \partial\Omega_t$$

where the vector $J_a$ is given by the formula

$$J_a = \lim_{|\Gamma_t| \Rightarrow 0} \int_{\Gamma_t} (-\sigma_{cb} w_{c,a} \kappa_b + \rho_0 f \kappa_a) \, dS$$

At every fixed time t the problem (5.22) without the boundary conditions at the crack tip can be obtained from the inequalities (5.18)-(5.20) and the following variational inequality

$$\delta_x \mathcal{E} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}[v_a(X_a, t, \varepsilon)] \geq 0 \quad , \quad \forall \; v_a(X_a, t, \varepsilon) \in \mathcal{C}_{\Omega_t} \tag{5.23}$$

One can also pass to the limit case of an elastic-perfectly-plastic cracked body by letting the hardening coefficients approach zero.

3. Geometrically linear dynamics of an elastic cracked body.

The law of motion of an elastic cracked body is given by

$$x_a = X_a + w_a(X_a, \tau) \quad , \quad X_a \in V_\tau = V \setminus (\Omega_\tau \cup \partial\Omega_\tau) \tag{5.24}$$

We suppose that $w_a \in \mathcal{C}_{\Omega_t} = \{ \, v_a \mid v_a$ and $v_{a,b} \in (H_t^1)^3 \; , \; (v_a^+ - v_a^-) N_a \geq 0$ on $\Omega_t \, \}$, $\dot{w}_a \in (L_t^1)^3$. The dot over quantities denotes their time derivative with $X_a$ held constant. Let $\mathcal{C}_t$ be the set of admissible comparison displacements

$$\mathscr{E}_\tau = \bigcup_{\Sigma_\tau \supseteq \Omega_\tau} \mathscr{E}_{\Sigma_\tau}$$

On $\mathscr{E}_t$ we define the energy functional of an elastic cracked body by the formula

$$\mathscr{E}[v_a(X_a,t)] = \int_{V_{\Sigma_t}} \rho_0 u(\varepsilon_{ab}(v))dX + \int_{\Sigma_t} 2\gamma\, dA + \int_{V_{\Sigma_t}} \rho_0 F_a v_a\, dX - \int_{\partial V_T} T_a v_a dA \qquad (5.25)$$

We introduce also the set $\mathcal{P}_t$ of parametrizations $Y_a(X_a,t)$ transforming $V_{\Omega_t}$ into $V_\Sigma$ , $\Omega_t$ into $\Sigma$ and satisfying the condition

$$Y_a(X_a,t) = X_a \quad \text{when } X_a \in \partial V$$

The geometrically linear dynamical crack problem is formulated in the following way:

Find $w_a \in \mathscr{E}_t$ such that the inequality

$$\delta\mathscr{E} + \int_{V_t} \rho_0 \ddot{w}_a(\delta v_a - w_{a,b}\delta Y_b)\, dX - \int_{\partial\Omega_t} Q_a \delta Y_a\, dS \geq 0 \qquad (5.26)$$

holds for arbitrary families of admissible comparison displacements $v_a(X_a,t,\varepsilon) \in \mathscr{E}_t$ and parametrizations $Y_a(X_a,t,\varepsilon) \in \mathcal{P}_t$. The definition of variation is exactly the same with that of the chapter 4. The vector $Q_a$ is given by

$$Q_a = \lim_{|\Gamma_t| \Rightarrow 0} \int_{\Gamma_t} \frac{1}{2} \rho_0 \dot{w}_b \dot{w}_b \kappa_a dS$$

We demand also the absolute equality from (5.26), if the real displacement field $w_a(X_a,\tau) = w_a(X_a,t+\varepsilon)$ is chosen for the family of admissible comparison displacements.

From this formulation one obtains the following consequences

$$\sigma_{ab,b} + \rho_0 F_a = \rho_0 \ddot{w}_a \quad , \quad \sigma_{ab} = \rho_0 \frac{\partial u}{\partial \varepsilon_{ab}}$$

$$\varepsilon_{ab} = \frac{1}{2}(w_{a,b} + w_{b,a})$$

$$w_a = 0 \quad \text{on } \partial V_x \quad , \quad \sigma_{ab} N_b = T_a \quad \text{on } \partial V_T$$

$$(w_a^+ - w_a^-)N_a \ge 0 \quad \text{on } \Omega_t \tag{5.27}$$

$$\sigma^+_{ab}N_b = \sigma^-_{ab}N_b = -pN_a \ , \ p\ge 0 \quad \text{on } \Omega_t$$

$$(w_a^+ - w_a^-)N_a > 0 \ \Rightarrow \ p = 0$$

$$|I_\alpha| = \sqrt{I_a I_a} \le 2\gamma \ , \qquad I_3 = I_a\tau_a = 0 \quad \text{on } \partial\Omega_t$$

$$\left.\begin{aligned} p > 0 \ &\Rightarrow \ [\dot{w}_a^+ - \dot{w}_a^-]\, N_a = 0 \\ p = 0 \ &\Rightarrow \ [\dot{w}_a^+ - \dot{w}_a^-]\, N_a \ge 0 \end{aligned}\right\} \text{on } \Omega_t$$

$$\left.\begin{aligned} |I_\alpha| < 2\gamma \ &\Rightarrow \ \dot{X}'_a = 0 \quad \text{(no propagation)} \\ |I_\alpha| = 2\gamma \ &\Rightarrow \ \dot{X}'_a\lambda_a \ge 0 \ , \quad 2\gamma\lambda_a = I_a \end{aligned}\right\} \text{on } \partial\Omega_t$$

where the flux of total energy entering into the crack tip is given by

$$I_a = \lim_{|\Gamma_t|\Rightarrow 0} \int_{\Gamma_t} \left[-\sigma_{cb}w_{c,a}\kappa_b + \rho_0(u + \frac{1}{2}\dot{w}_b\dot{w}_b)\kappa_a\right]dS \tag{5.28}$$

4. Geometrically linear dynamics of an elastic-plastic cracked body.

In this theory one should determine the displacement field $w_a(X_a,t)\in \mathscr{E}_{\Omega_t}$ of (5.24), the internal degrees of freedom $\varepsilon^p_{ab}(X_a,\tau)\in(L^2_\tau)^6$, $\chi_A(X_a,\tau)\in(L^2_\tau)^N$ and the unknown interior boundary $\Omega_\tau U\partial\Omega_\tau$.

We define the energy functional of an elastic-plastic cracked body on the set $\mathscr{E}_t$ of admissible comparison displacements as follows

$$\mathscr{E}[v_a(X_a,\tau)] = \int_{V_{\Sigma_\tau}} \rho_0 u(\varepsilon_{ab}(v)-\varepsilon^p_{ab},\chi_A)dX + \int_{\Sigma_\tau} 2\gamma dA + \int_{V_{\Sigma_\tau}} \rho_0 F_a v_a dX - \int_{V_T} T_a v_a dA \tag{5.29}$$

Now we can formulate the geometrically linear dynamical problem for an elastic -plastic cracked body:

Find $w_a\in \mathscr{E}_t$ , $\varepsilon^p_{ab} \in (L^2_t)^6$ , $\chi_A \in (L^2_t)^N$ such that the inequalities

$$\mathscr{F}\left( \rho_0\frac{\partial u}{\partial\varepsilon_{ab}} \ , \ -\rho_0\frac{\partial u}{\partial\chi_A} \right) \le 0$$

$$(\hat{\sigma}_{ab} - \rho_0\frac{\partial u}{\partial\varepsilon_{ab}})\ \dot{\varepsilon}^p_{ab} + (\hat{\pi}_A + \rho_0\frac{\partial u}{\partial\chi_A})\ \dot{\chi}_A \le 0 \ , \quad \forall\ (\hat{\sigma}_{ab},\hat{\pi}_A) \tag{5.30}$$

$$\mathcal{F}(\hat{\sigma}_{ab},\hat{\pi}_A) \le 0$$

$$\delta_x\mathcal{E} + \int_{V_t} \rho_0\ddot{w}_a(\delta v_a - w_{a,b}\delta Y_b)\ dX - \int_{\partial\Omega_t} Q_a\delta Y_a\ dS \ge 0$$

hold for arbitrary $v_a(X_a,t,\varepsilon)\in \mathcal{E}_t$ and $Y_a(X_a,t,\varepsilon)\in \mathcal{P}_t$ . Moreover, the last inequality of (5.30) should turn into the absolute equality if the real displacement field $w_a(X_a,\tau) = w_a(X_a,t+\varepsilon)$ is chosen for the family of admissible comparison displacements.

From this formulation one obtains the folowing consequences

$$\mathcal{F}(\sigma_{ab},\pi_A) \le 0$$

$$(\hat{\sigma}_{ab}-\sigma_{ab})\dot{\varepsilon}^P_{ab} + (\hat{\pi}_A-\pi_A)\dot{\chi}_A \le 0 \quad , \quad \forall\ (\hat{\sigma}_{ab},\hat{\pi}_A),\quad \mathcal{F}(\hat{\sigma}_{ab},\hat{\pi}_A) \le 0$$

$$\sigma_{ab,b} + \rho_0 F_a = \rho_0\ddot{w}_a$$

$$\sigma_{ab} = \rho_0\,\frac{\partial u}{\partial \varepsilon_{ab}} \quad , \quad \pi_A = -\rho_0\,\frac{\partial u}{\partial \chi_A}$$

$$\varepsilon_{ab} = \frac{1}{2}\,(w_{a,b} + w_{b,a})$$

$$w_a = 0 \quad \text{on } \partial V_x \quad , \quad \sigma_{ab}N_b = T_a \quad \text{on } \partial V_T$$

$$(w^+_a - w^-_a)N_a \ge 0 \quad \text{on } \Omega_t \tag{5.31}$$

$$\sigma^+_{ab}N_b = \sigma^-_{ab}N_b = -pN_a \ ,\ p\ge 0 \quad \text{on } \Omega_t$$

$$(w^+_a - w^-_a)N_a > 0 \ \Rightarrow\ p = 0$$

$$|I_\alpha| = \sqrt{I_a I_a} \le 2\gamma \ , \quad I_3 = I_a\tau_a = 0 \quad \text{on } \partial\Omega_t$$

$$\left.\begin{array}{l} p > 0 \ \Rightarrow\ [\dot{w}^+_a - \dot{w}^-_a]\ N_a = 0 \\ p = 0 \ \Rightarrow\ [\dot{w}^+_a - \dot{w}^-_a]\ N_a \ge 0 \end{array}\right. \quad \text{on } \Omega_t$$

$$\left.\begin{array}{l} |I_\alpha| < 2\gamma \ \Rightarrow\ \dot{X}'_a = 0 \quad \text{(no propagation)} \\ |I_\alpha| = 2\gamma \ \Rightarrow\ \dot{X}'_a\lambda_a \ge 0\ , \quad 2\gamma\lambda_a = I_a \end{array}\right. \quad \text{on } \partial\Omega_t$$

where the vector $I_a$ is given by the formula (5.28).

## 6. Examples.

We shall now consider some simple applications of the developed theory.

1. The finite anti-plane shear of an infinite incompressible elastic slab with a flat gap.

Suppose an isotropic homogeneous incompressible elastic slab occupies the infinite cylindrical region $\mathbb{R}^2_\Omega x\mathbb{R} = (\mathbb{R}^2\backslash\bar{\Omega})x\mathbb{R}$ in the initial state, where $\Omega$ is a line-segment of length 2a (Fig.7). Consider the deformation

$$x_i = x_i(X_a) \quad , \quad X_a \in \mathbb{R}^2_\Omega x\mathbb{R} \tag{6.1}$$

Incompressibility condition requires that

$$\det|F| \equiv 1 \ , \ F_{ia} = x_{i,a} \quad , \quad \forall\ X_a \in \mathbb{R}^2_\Omega x\mathbb{R} \tag{6.2}$$

Supposing the simplest case of neo-Hookean materials [55], one has

$$\rho_0 f = \mu\varepsilon_{aa} = \frac{\mu}{2}(x_{i,a}x_{i,a} - 3) \tag{6.3}$$

where $\mu>0$ is the (constant) shear modulus. From (6.3) and (6.2) follows

$$T_{ia} = \mu x_{i,a} - qG_{ia} \quad ; \quad G = (F^T)^{-1} \tag{6.4}$$

In the absence of mass force, the basic equations of equilibrium for such the infinite slab are (cf. the formulae (2.15),(2.20)-(2.23))

$$T_{ia,a} = 0 \quad , \quad T_{ia} = \mu x_{i,a} - qG_{ia} \tag{6.5}$$

$$\det|x_{i,a}| = 1 \tag{6.6}$$

$$T^{\mp}_{ia}N_a = 0 \text{ on } (\Omega\backslash\Omega^{\mp})x\mathbb{R} \tag{6.7}$$

$$T^{+}_{i2}\sqrt{A/a}\ \Big|_{\eta_\alpha} = T^{-}_{i2}\sqrt{A/a}\ \Big|_{\theta_\alpha} = -pn_i \ , \ p\geq 0 \quad \text{on } \Omega^{+}x\mathbb{R} \tag{6.8}$$

$$|J_\alpha| = \sqrt{J_\alpha J_\alpha} \leq 2\gamma \text{ on } \partial\Omega x\mathbb{R} \tag{6.9}$$

Here the J-integrals are given by

$$J_a = \lim_{|\Gamma|\Rightarrow 0} \int_\Gamma (-T_{i\beta}x_{i,\alpha}\kappa_\beta + \rho_0 f\kappa_\alpha)\ dS \tag{6.10}$$

The contour of integration should lie in the plane $\mathbb{R}^2$ , surrounding the points $(\mp a,0)$. For the exterior boundary condition instead of (2.17) one has to specify the asymptotic displacement field at infinity. We suppose, that the displacement field approaches that of a simple shear parallel to the gap surface

$$x_1 = X_1 \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3 + kX_2 \quad \text{as } X_1^2 + X_2^2 \Rightarrow \infty \tag{6.11}$$

Due to this condition, one may seek the solution of (6.5)-(6.9) in the form

$$x_1 = X_1 \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3 + w(X_1,X_2) \tag{6.12}$$

For the deformation (6.12) the deformation gradients have the form

$$x_{i,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_{,1} & w_{,2} & 1 \end{pmatrix} \quad , \quad G_{ia} = \begin{pmatrix} 1 & 0 & -w_{,1} \\ 0 & 1 & -w_{,2} \\ 0 & 0 & 1 \end{pmatrix} \tag{6.13}$$

One can see that the incompressibility condition (6.6) is automatically satisfied. It then follows from (6.5),(6.11) that the components $T_{ia}$ are given by

$$T_{\alpha\beta} = (\mu-q)\delta_{\alpha\beta} \quad , \quad T_{\alpha 3} = qw_{,\alpha}$$

$$T_{3\alpha} = \mu w_{,\alpha} \quad , \quad T_{33} = \mu-q \tag{6.14}$$

Substitution from (6.14) into the equilibrium equations (6.5) reduces these to

$$-q_{,\alpha} + q_{,3}w_{,\alpha} = 0 \quad , \quad \mu w_{,\alpha\alpha} = q_{,3} \tag{6.15}$$

Since the left hand side of the second equation in (6.15) is independent of $X_3$, it follows that q is linear in $X_3$. The first of (6.15) then requires that $q_{,\alpha}$ be independent of $X_3$, so that

$$q(X_1,X_2,X_3) = d_0X_3 + q_1(X_1,X_2) \tag{6.16}$$

Substitution from (6.16) into the first of (6.15) gives

$$q_1 = \mu + d_0w + d_1 \tag{6.17}$$

where $d_1$ is an arbitrary constant.

The first Piola-Kirchhoff stress tensor in (6.14) may be reduced with the help of (6.16),(6.17) to

$$T_{33} = -[d_0(X_3+w) + d_1]$$

$$T_{3\alpha} = \mu w_{,\alpha} \quad , \quad T_{\alpha 3} = [\mu + d_0(X_3+w) + d_1] \; w_{,\alpha} \tag{6.18}$$

$$T_{\alpha\beta} = -[d_0(X_3+w) + d_1] \; \delta_{\alpha\beta}$$

The only remaining differential equation of equilibrium is the second of (6.15), which is now

$$\mu \; \nabla . \nabla \; w = d_0 \quad , \quad \nabla . \nabla = \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} \tag{6.19}$$

Now we intend to determine $d_0$ , $d_1$ from boundary conditions. In the case of simple shear, the contact of the gap banks takes place and $\Omega^+=\Omega^-=\Omega$. But one can show that p=0 on $\Omega$, so the gap surfaces are traction-free

$$T^{\mp}_{12} = T^{\mp}_{22} = T^{\mp}_{32} = 0 \quad \text{for } X_2 = 0 \; , \; |X_1| < a \tag{6.20}$$

Reference to (6.18) shows that the first condition in (6.20) is automatically satisfied, while the second requires that

$$d_0 = d_1 = 0 \tag{6.21}$$

Because of (6.21), one can reduce the first Piola-Kichhoff stress tensor to the form

$$T_{3\alpha} = T_{\alpha 3} = \mu \; w_{,\alpha} \quad , \quad T_{\alpha\beta} = 0 \quad , \quad T_{33} = 0 \tag{6.22}$$

The third of the conditions (6.20) leads to

$$w^{\mp}_{,2} = 0 \quad \text{at } X_2 = 0 \; , \; |X_1| < a \tag{6.23}$$

In view of (6.21), the differential equation (6.19) for w is

$$\nabla . \nabla \; w = 0 \quad \text{in } \mathbb{R}^2_\Omega \tag{6.24}$$

According to (6.11), at infinity w should satisfy the following asymptotic condition

$$w = kX_2 + o(1) \quad \text{as } X_1^2 + X_2^2 \Rightarrow \infty \tag{6.25}$$

The linear boundary-value problem (6.23)-(6-25) is mathematically identical to the problem of steady irrotational flow of an inviscid, incompressible fluid past a flat plate of width 2a at an angle of attack of $90^{\circ}$. By this analogy, one can immediately find the solution w and the shear stress $T_{\alpha 3}$. If r, $\vartheta$ are polar co-ordinates at the right gap tip (see Fig.7), so that

$$X_1 = a + r\cos\vartheta \quad , \quad X_2 = r\sin\vartheta \tag{6.26}$$

one can readily show that the asymptotic behaviour of w and $T_{\alpha 3}$ near the gap tip is given by

$$w = k\ (2ar)^{1/2}\sin\frac{\vartheta}{2} \tag{6.27}$$

$$T_{13} = T_{31} = -\mu ka\ (2ar)^{1/2}\sin\frac{\vartheta}{2} \quad , \quad T_{23} = T_{32} = \mu ka\ (2ar)^{1/2}\cos\frac{\vartheta}{2}$$

as $r \Rightarrow 0$.

Using the formulae (6.10),(6.27) one can easily calculate $J_\alpha$. At the right gap tip $J_\alpha$ is given by

$$J_1 = \frac{1}{2}\,\mu k^2 a\pi \quad , \quad J_2 = 0 \quad \text{for } X_2=0\ ,\ X_1=a \tag{6.28}$$

Due to the symmetry, one has

$$J_1 = -\frac{1}{2}\,\mu k^2 a\pi \quad , \quad J_2 = 0 \quad \text{for } X_2=0\ ,\ X_1=-a \tag{6.29}$$

From (6.9) and (6.28),(6.29) one concludes that only slabs, containing a gap of width less than, or equal to, $8\gamma/(\mu k^2\pi)$ can be in the state of equilibrium under an anti-plane shear.

2. The small plane deformation of an infinite elastic body with an angled gap.

Suppose an isotropic homogeneous elastic body occupies the infinite cylindrical region $\mathbb{R}^2_\Omega \times \mathbb{R}$ in the initial state, where $\Omega$ is a line segment of length 2a, oriented at an angle $\beta$ to the direction of the tensile stress $\bar{\sigma}$ (Fig.8). On supposing the small plane deformation, one seeks the equilibrium displacement field in the form

$$w_\alpha = w_\alpha(X_1,X_2) \quad , \quad w_3 \equiv 0 \quad , \quad X_\alpha \in \mathbb{R}^2_\Omega \tag{6.30}$$

The free energy density and the constitutive relation are given by

$$\rho_0 f = \frac{1}{2}\lambda\,(\varepsilon_{cc})^2 + \mu\,\varepsilon_{ab}\varepsilon_{ab}$$

$$\sigma_{ab} = \lambda\,\varepsilon_{cc}\delta_{ab} + 2\,\mu\,\varepsilon_{ab} \tag{6.31}$$

Taking into account (6.30),(6.31) and the absence of mass force, we reduce the system of equations (5.12) to

$$\sigma_{\alpha\beta,\beta} = 0 \quad , \quad \sigma_{\alpha\beta} = \lambda\,\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\,\mu\,\varepsilon_{\alpha\beta}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\,(w_{\alpha,\beta} + w_{\beta,\alpha})$$

$$\sigma_{\alpha\beta} \simeq \bar{\sigma}\,\delta_{\alpha2}\delta_{\beta2} \quad \text{as } X_1^2 + X_2^2 \Rightarrow \infty$$

$$(w_\alpha^+ - w_\alpha^-)\,N_\alpha \geq 0 \quad \text{on } \Omega \tag{6.32}$$

$$\sigma_{\alpha\beta}^+ N_\beta = \sigma_{\alpha\beta}^- N_\beta = -p\,N_\alpha \quad , \quad p \geq 0 \quad \text{on } \Omega$$

$$(w_\alpha^+ - w_\alpha^-)\,N_\alpha > 0 \;\Rightarrow\; p = 0 \quad \text{on } \Omega \;.$$

$$|J_\alpha| = \sqrt{J_\alpha J_\alpha} \leq 2\gamma \text{ on } \partial\Omega$$

Here

$$\Omega = \{\,(X_1,X_2)\;|\;X_1 = \mathrm{tg}\,\beta\,X_2 \quad , \quad |X_2| \leq a\cos\beta\,\}$$

$$N_\alpha = \{\,-\cos\beta\;,\;\sin\beta\,\}$$

$$J_\alpha = \lim_{|\Gamma| \Rightarrow 0} \int_\Gamma (-\sigma_{\gamma\beta} w_{\gamma,\alpha}\kappa_\beta + \rho_0 f \kappa_\alpha)\,dS$$

In the case of tensile stress ($\bar{\sigma} > 0$), one can easily show that $p \equiv 0$, so the gap surface $\Omega^{\mp}$ are traction free. One seeks the solution of (6.32) in the form

$$\sigma_{\alpha\beta} = \bar{\sigma}\,\delta_{\alpha2}\delta_{\beta2} + \sigma'_{\alpha\beta} \tag{6.33}$$

$$w_\alpha = \frac{\bar{\sigma}}{\lambda + 2\mu}\,\delta_{\alpha2}X_2 + w'_\alpha$$

Due to the linearity the equations for $\sigma'_{\alpha\beta}$ , $w'_\alpha$ take the form

$$\sigma'_{\alpha\beta,\beta} = 0 \quad , \quad \sigma'_{\alpha\beta} = \lambda\, \varepsilon'_{\gamma\gamma}\delta_{\alpha\beta} + 2\,\mu\,\varepsilon'_{\alpha\beta}$$

$$\varepsilon'_{\alpha\beta} = \frac{1}{2}\,(w'_{\alpha,\beta} + w'_{\beta,\alpha}) \tag{6.34}$$

$$\varepsilon'_{\alpha\beta} \simeq 0 \quad \text{as } X_1^2 + X_2^2 \Rightarrow \infty$$

$$\sigma'^{+}_{\alpha\beta}\, N_\beta = \sigma'^{-}_{\alpha\beta}\, N_\beta = -\bar{\sigma}\sin\beta\;\delta_{\alpha 2} \quad \text{on } \Omega$$

The problem (6.34) itself can be decomposed into two problem as it is shown in Fig.9. Using the method of complex potential [21], one can obtain the solution of (6.34). If $r,\vartheta$ are polar co-ordinates at the right gap tip (see Fig.7), then the asymptotic behaviour of $w'_\alpha$ and $\sigma'_{\alpha\beta}$ near the gap tip is given by

$$\sigma'_{11} = \frac{K_1}{(2\pi r)^{1/2}}\cos\frac{\vartheta}{2}\left(1-\sin\frac{\vartheta}{2}\sin\frac{3}{2}\vartheta\right) - \frac{K_2}{(2\pi r)^{1/2}}\sin\frac{\vartheta}{2}\left(2+\cos\frac{\vartheta}{2}\cos\frac{3}{2}\vartheta\right)$$

$$\sigma'_{22} = \frac{K_1}{(2\pi r)^{1/2}}\cos\frac{\vartheta}{2}\left(1+\sin\frac{\vartheta}{2}\sin\frac{3}{2}\vartheta\right) + \frac{K_2}{(2\pi r)^{1/2}}\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\cos\frac{3}{2}\vartheta$$

$$\sigma'_{12} = \frac{K_1}{(2\pi r)^{1/2}}\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\cos\frac{3}{2}\vartheta - \frac{K_2}{(2\pi r)^{1/2}}\cos\frac{\vartheta}{2}\left(1-\sin\frac{\vartheta}{2}\sin\frac{3}{2}\vartheta\right) \tag{6.35}$$

$$w'_1 = \frac{K_1}{\mu}\left(\frac{r}{2\pi}\right)^{1/2}\cos\frac{\vartheta}{2}\left(1-2\nu+\sin^2\frac{\vartheta}{2}\right) + \frac{K_2}{\mu}\left(\frac{r}{2\pi}\right)^{1/2}\sin\frac{\vartheta}{2}\left(2-2\nu+\cos^2\frac{\vartheta}{2}\right)$$

$$w'_2 = \frac{K_1}{\mu}\left(\frac{r}{2\pi}\right)^{1/2}\sin\frac{\vartheta}{2}\left(2-2\nu-\cos^2\frac{\vartheta}{2}\right) - \frac{K_2}{\mu}\left(\frac{r}{2\pi}\right)^{1/2}\cos\frac{\vartheta}{2}\left(2-2\nu-\sin^2\frac{\vartheta}{2}\right)$$

where $\nu$ is the Poisson's coefficient : $\nu = \lambda/2(\lambda+\mu)$. The stress intencity factors $K_1$ , $K_2$ are (see for example [40])

$$K_1 = \bar{\sigma}\,\sqrt{\pi a}\,\sin^2\beta \quad , \quad K_2 = \bar{\sigma}\,\sqrt{\pi a}\,\sin\beta\,\cos\beta \tag{6.36}$$

Let us find the flux of energy $J_\alpha$ . Denote by **i** and **j** the unit vectors of the local system of co-ordinates associated with the gap

$$\mathbf{i} = \{\, \sin\beta \;,\; \cos\beta \,\} \quad , \quad \mathbf{j} = \{\, -\cos\beta \;,\; \sin\beta \,\}$$

By long but elementary calculations of trigonometric functions of the form (6.35), one can show that (see also [56])

$$J = \frac{1-\nu}{2\mu}\,[(K_1^2 + K_2^2)i - 2K_1K_2j] \tag{6.37}$$

By using (6.36) and (6.37) we rewrite explicitly the equilibrium condition in the form

$$|J| = \frac{1-\nu}{2\mu}\,[(K_1^2 + K_2^2)^2 + 4K_1^2K_2^2]^{1/2} =$$

$$= \frac{1-\nu}{2\mu}\,\bar{\sigma}^2\pi a \sin^2\beta\,(1+\sin^2 2\beta)^{1/2} \le 2\,\gamma$$

In Fig.10 the region of dimensionless stress $\mathcal{P} = \bar{\sigma}/2[(1-\nu)\pi a/\mu\gamma]^{1/2}$ for which the equilibrium state is possible is shown in dependence with the angle $\beta$. It is interesting to note, that the critical tensile stress will have the minimum value not for the angle $\beta=90^\circ$, but for the angle $\beta\simeq71^\circ$, for which $\mathcal{P}\simeq0,976$.

One can also calculate the angle $\alpha$ of initial crack propagation defined by

$$\mathrm{tg}\,\alpha = J_2/J_1 = -\sin 2\beta$$

The graphic of $\alpha$ in dependence of $\beta$ is shown in the Fig.11.

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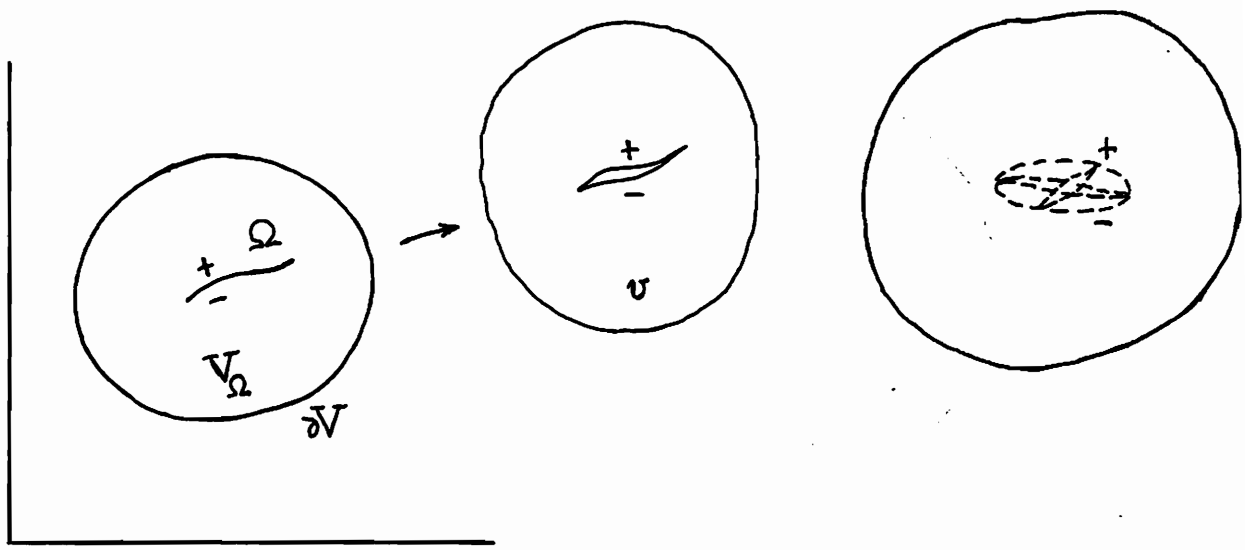


Fig.1: Deformation of gapped body.

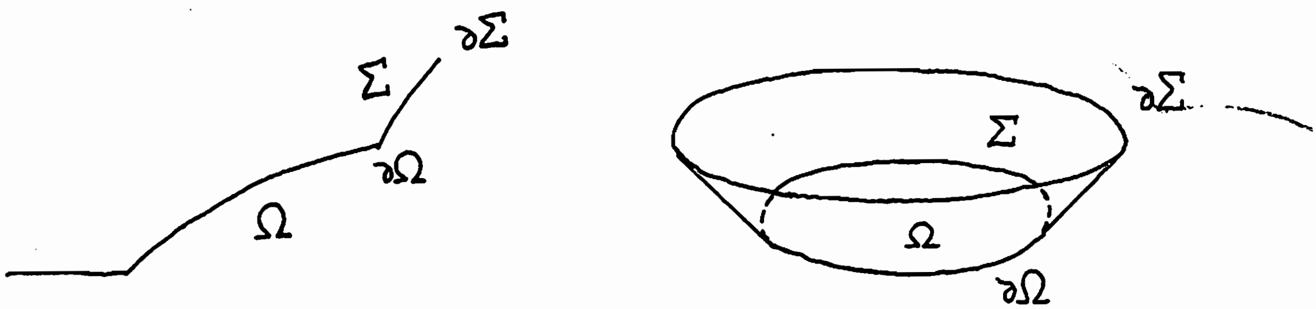


Fig.2: Discontinuity surface of admissible comparison configuration

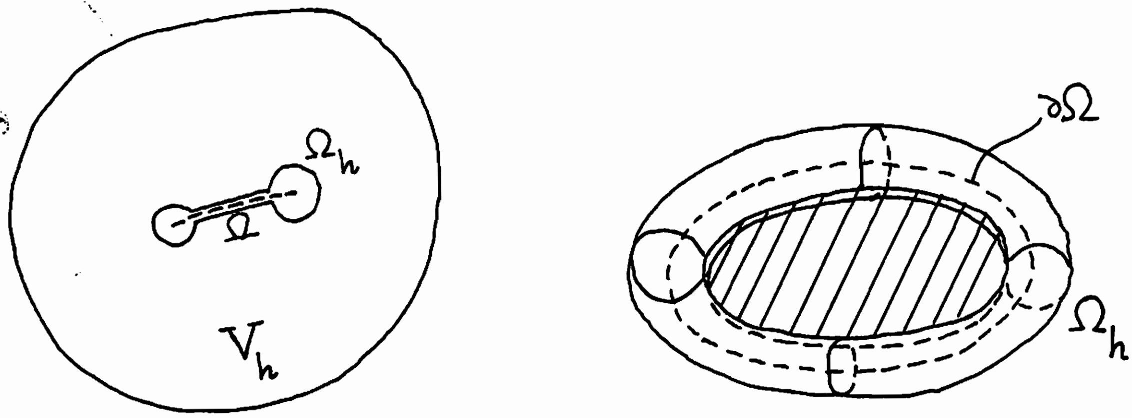


Fig.3: Region $V_h$ and interior boundary $\Omega_h$ .

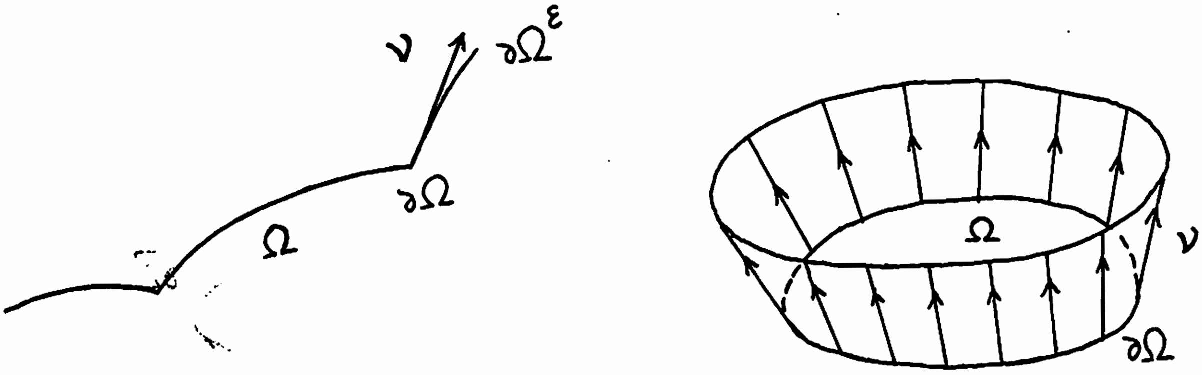


Fig.4: Vector **ν** .

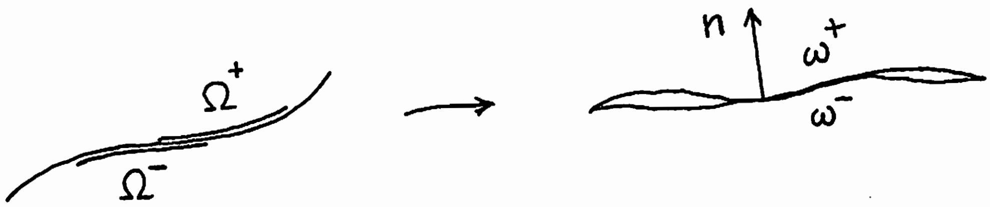


Fig.5: Contact surfaces.

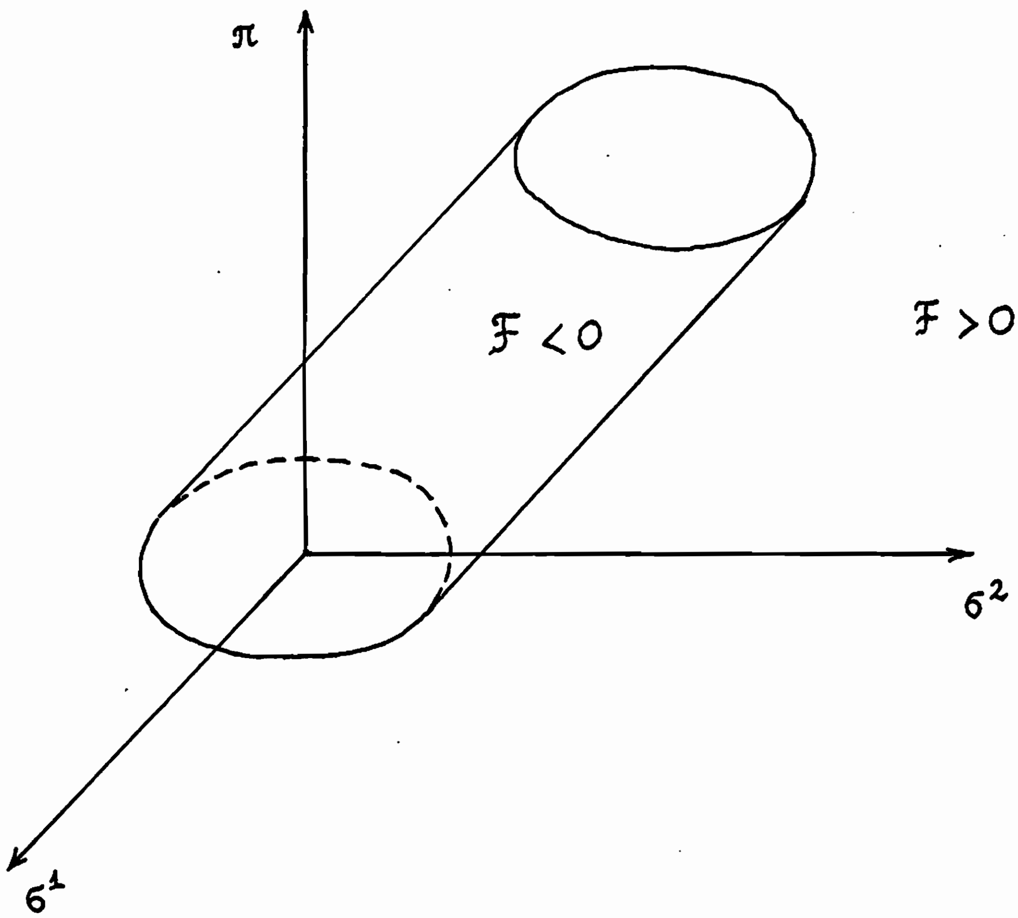


Fig.6: Yield function.

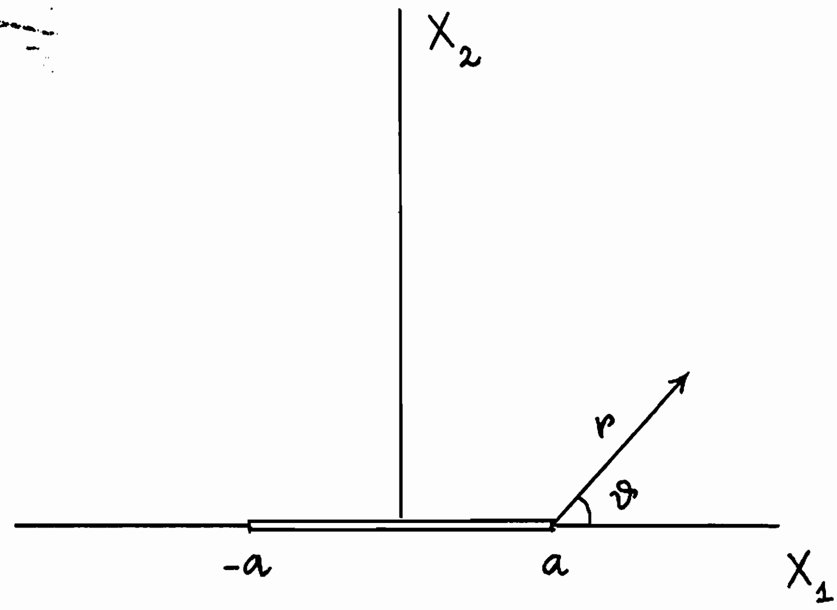


Fig.7: Geometry of first problem.

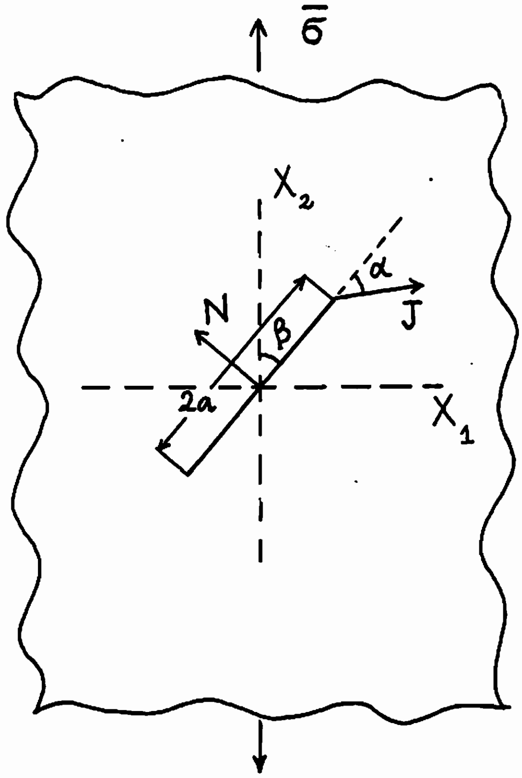


Fig.8: Geometry of second problem.

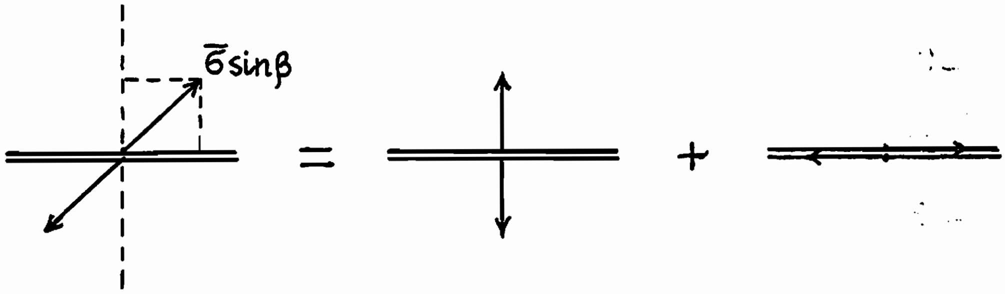


Fig.9: Decomposition of second problem.

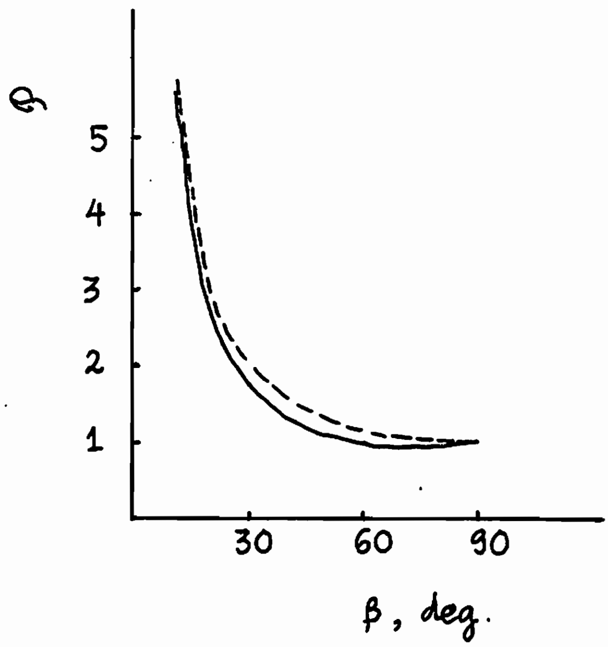


Fig.10: Equilibrium tensile stress.

----: Irwin's theory.

——: Present theory.

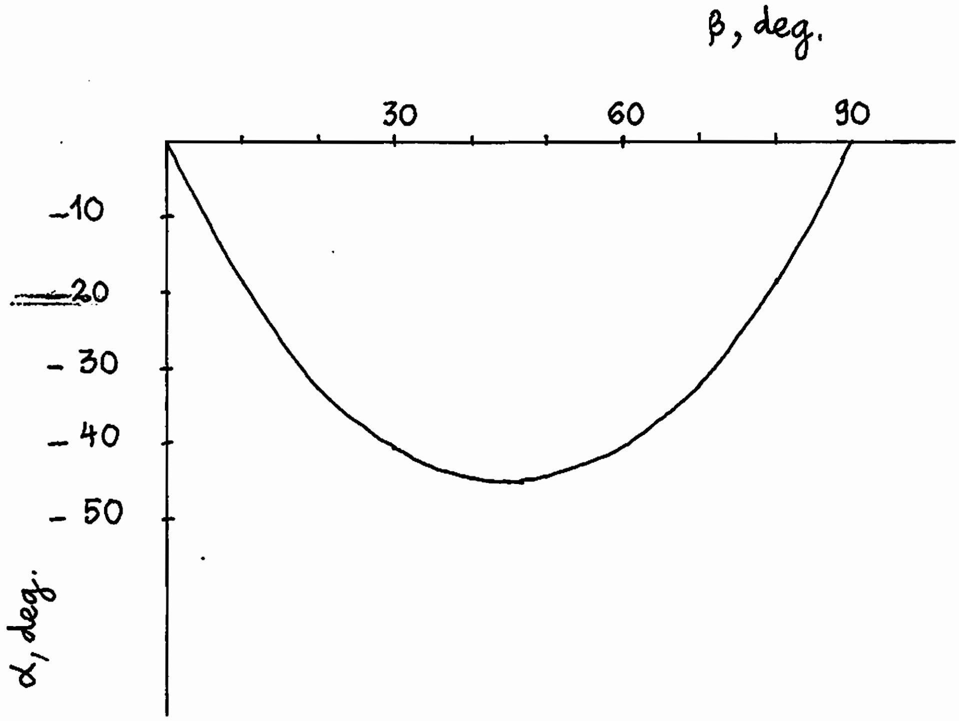


Fig. 11: Function $\alpha(\beta)$.

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