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Foundations of the Non-Linear Plastic Shell Theory

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SUMMARY

The paper contains the formulation of basic relations of the geometrically non-linear plastic shell theory illustrated by simple examples of application.

The theory is developed within the framework of simplifying assumptions which are less restrictive than the assumptions of the Donnell-Mushtari-Vlasov theory.

A classification of non-linear shell theories with regard to the initial shape of the shell and the deformation mode is proposed. The method of simplifying the geometrical relations is presented and a number of approximate theories are formulated.

Yield surfaces for uniform and sandwich shells are considered using the Huber-Mises and the Tresca conditions.

The influence of geometric changes on the load carrying capacity is illustrated with the examples of cylindrical and shallow sperical shells. The attention is focused on the behaviour at moderately large deflections.

The elaboration contains a number of original results, particularly in the chapters five and ten.

ZUSAMMENFASSUNG

Die Arbeit beinhaltet die Formulierung grundlegender Beziehungen der geometrisch nicht-linearen plastischen Schalentheorie, veranschaulicht durch einfache Anwendungsbeispiele. Die Theorie ist in einem Rahmen entwickelt, dessen vereinfachende Annahmen weniger einschränkend sind als die Voraussetzungen der Donnell-Mushtari-Vlasov-Theorie.

Eine Klassifizierung nicht-linearer Schalentheorien unter Bezug auf die ursprüngliche Form der Schale sowie des Verformungscharakters wird vorgeschlagen. Die Vereinfachungsmethode für die geometrischen Beziehungen wird vorgestellt und eine Anzahl von Approximationstheorien wird formuliert.

Fließflächen für uniforme sowie Sandwich-Schalen werden unter Benutzung der Huber-Mises- und der Tresca-Bedingungen betrachtet.

Der Einfluß von Geometrieänderungen auf die Tragfähigkeit wird durch die Beispiele zylinderischer und flacher sphärischer Schalen illustriert. Hauptaugenmerk wird auf das Verhalten bei moderierten Durchbiegungen gelegt.

Die Ausarbeitung enthält eine Anzahl originaler Ergebnisse, insbesondere in den Kapiteln fünf und zehn.

CONTENTS

1. INTRODUCTION

Over the recent years the study of the plastic behaviour of thin-walled structures has become an area of increasing importance. There are numerous practical situations in which shells are stressed beyond the elastic limit of the material. The elastic strains remain usually infinitesimal (though elastic deflections may be finite) so the elastic-plastic or rigid-plastic models of material can be applied. Such structures will generally undergo large plastic deformations prior to failure. Therefore the influence of geometry changes on their ultimate load carrying capacity must be taken into consideration.

A geometrically nonlinear behaviour can be studied employing either the material or the spatial description, thus refering the process of deformation to the initial or to the actual configuration, respectively. The first alternative, i. e. the Lagrangian description seems to be preferable, as applied to the thin-walled structures, since the boundary conditions are usually referred to the undeformed configuration of the structure and because the material rate of change of stress and strain rate tensors are invariant with respect to the rigid body motion.

An important problem in the theory of plastic structures is that of the criterion of yielding. In the theory of plasticity the yield criteria are usually written in terms of the true stress components referred to the actual configuration; whereas the shell theory is formulated in terms of the stress resultants rather than the stress components themselves. A necessity thus arises to transform the yield condition into the space of stress resultants in the suitably chosen reference configuration.

The behaviour of a plastic shell is governed by the following system of equations:

a) kinematical relations specifying the deformation (extension and curvature) of the middle surface in terms of the displacements and displacement gradients,

b) motion equations expressed in terms of the stress resultants and stress couples,

c) boundary conditions,

d) a yield criterion saying at what combination of stress resultants the shell wall cross-section may yield,

e) a flow law specifying the relationship between the stress resultants and the strain rates, to predict how an element of a shell structure will deform when the yield condition is reached.

The load-carrying capacity assessed by the tools of limit analysis simply indicates a load intensity at which plastic deformations begin to develop. If changes in geometry could be neglected, the plastic deformations of a perfectly plastic structure would develop under constant loads. If, however, those changes are taken into account, it may be found that plastic deformation continues to develop under either increasing or constant or decreasing loads. In the last two cases the load-carrying capacity represents a critical load intensity which must not be exceeded if the structure is to survive. Therefore, full appreciation of the physical significance of the load-carrying capacity furnished by limit analysis usually requires the analysis of the behaviour of the structure after the yield-point load has been reached.

The limit analysis methods of determination of the load-carrying capacity and investigation of the post yield behaviour of plastic shells will be presented and discussed. Attention will be focused on the geometrically non-linear behaviour of thin shells made of rigid-perfectly plastic material.

Examples of cylindrical and spherical shells will be given.

2. GEOMETRIC PRELIMINARIES

In order to make the present paper reasonably self-contained, let us start with a brief summary of the description of a surface.

Let $\underset{\sim}{R}$ denotes the radius vector from a fixed origin O, in the Euclidean 3-space, to a generic point M on the reference surface "S" (Fig. 2.1). The radius $\underset{\sim}{R}$ will be treated as a vector function of the pair of Gaussian coordinates (X^1, X^2). The parametric lines X^Δ = const. form a net on the reference surface, and X^3 stands for a coordinate along the outward normal to the reference surface.

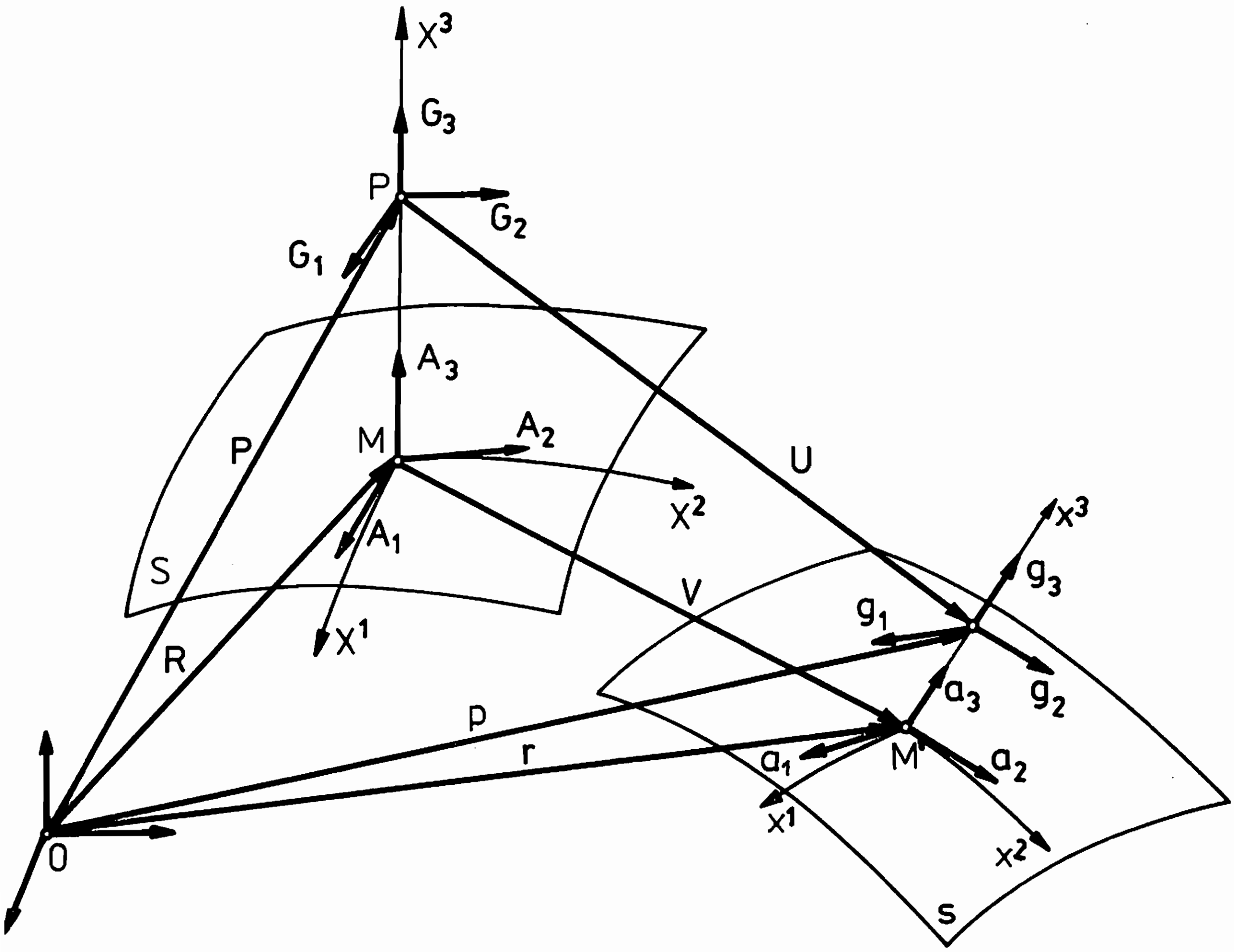

Fig. 2.1

Greek indices will always refer to the Gaussian coordinates and run over 1,2 ($\Delta,\Gamma,\alpha,\beta = 1,2$); and Latin indices will refer to the spatial coordinates and run over 1,2,3 ($K, L, i, j = 1,2,3$).

To distinguish between the initial and the deformed configurations, capital letters will be used for the former and lower case letters for the latter.

The position vector to the point M' on the deformed reference surface "s" will be denoted by $\underset{\sim}{r}(x^{\alpha})$ (Fig. 2.1).

The covariant base vectors $\underset{\sim}{A}_{\Delta}$, $\underset{\sim}{a}_{\alpha}$ at the generic points M, M' with coordinates X^{Δ} and x^{α} in undeformed and deformed configurations, respectively, are defined by

$$\underset{\sim}{A}_{\Delta} \equiv \underset{\sim}{R}_{,\Delta} \quad , \quad \underset{\sim}{a}_{\alpha} \equiv \underset{\sim}{r}_{,\alpha} \tag{2.1}$$

where comma followed by Δ or α indicates partial differentiation with respect to X^{Δ} or x^{α}, respectively.

The systems of the base vectors in the initial and in the deformed configurations are completed by the vectors $\underset{\sim}{A}_3$ and $\underset{\sim}{a}_3$. For the normal coordinate systems the vectors $\underset{\sim}{A}_3$ and $\underset{\sim}{a}_3$ coincide with the unit vectors $\underset{\sim}{N}$ and $\underset{\sim}{n}$ normal to the undeformed and deformed reference surfaces, respectively, so that

$$\underset{\sim}{A}_{\Delta} \cdot \underset{\sim}{A}_3 = 0 \quad , \quad \underset{\sim}{A}_3 \cdot \underset{\sim}{A}_3 = 1 \ , \tag{2.2}$$

$$\underset{\sim}{a}_{\alpha} \cdot \underset{\sim}{a}_3 = 0 \quad , \quad \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_3 = 1 \ . \tag{2.3}$$

The metric tensor (or the first fundamental tensor) of the undeformed and the deformed surfaces are defined by the scalar products of the respective base vectors

$$A_{\Delta\Gamma} \equiv \underset{\sim}{A}_{\Delta} \cdot \underset{\sim}{A}_{\Gamma} \quad , \quad a_{\alpha\beta} \equiv \underset{\sim}{a}_{\alpha} \cdot \underset{\sim}{a}_{\beta} \ . \tag{2.4}$$

The determinants of the metric tensors are denoted, respectively, by A and a

$$A \equiv |A_{\Delta\Gamma}| \quad , \quad a \equiv |a_{\alpha\beta}| \quad . \tag{2.5}$$

The reciprocal (contravariant) base vectors $\underset{\sim}{A}^{\Delta}$, $\underset{\sim}{a}^{\alpha}$ and the conjugate tensors $A^{\Delta\Gamma}$, $a^{\alpha\beta}$ are defined by

$$\underset{\sim}{A}^{\Delta} \cdot \underset{\sim}{A}_{\Gamma} = \delta^{\Delta}_{\Gamma} \quad , \quad A^{\Delta\Gamma} = \underset{\sim}{A}^{\Delta} \cdot \underset{\sim}{A}^{\Gamma} \quad , \tag{2.6}$$

$$\underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{a}_{\beta} = \delta^{\alpha}_{\beta} \quad , \quad a^{\alpha\beta} = \underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{a}^{\beta} \quad . \tag{2.7}$$

where δ^{Γ}_{Γ} and δ^{α}_{β} stand for the Kronecker delta.

The curvature (or second fundamental) tensors of the undeformed and the deformed surfaces can be defined by the scalar products

$$B_{\Delta\Gamma} = B_{\Gamma\Delta} \equiv A_3 \cdot A_{\Delta,\Gamma} = -\underset{\sim}{A}_{3,\Delta} \cdot \underset{\sim}{A}_{\Gamma} = -\underset{\sim}{A}_{3,\Gamma} \cdot \underset{\sim}{A}_{\Delta} \; , \tag{2.8}$$

$$b_{\alpha\beta} = b_{\beta\alpha} \equiv a_3 \cdot a_{\alpha,\beta} = -\underset{\sim}{a}_{3,\alpha} \cdot \underset{\sim}{a}_{\beta} = -\underset{\sim}{a}_{3,\beta} \cdot \underset{\sim}{a}_{\alpha} \; . \tag{2.9}$$

The third fundamental tensors of the undeformed and the deformed surfaces are defined, respectively, by

$$C_{\Delta\Gamma} \equiv B^{\phi}_{\Delta} B_{\phi\Gamma} \; , \tag{2.10}$$

$$c_{\alpha\beta} \equiv b^{\lambda}_{\alpha} b_{\lambda\beta} \; . \tag{2.11}$$

A point of a shell may be identified by the components X^K referred to the normal coordinate system in the Euclidean 3-space. The equation $X^3 = 0$ specifies the middle surface. The regions $|X^3| \leq H$, where $2H$ is the thickness of the shell, will be referred to as the shell space.

The position vectors $\underset{\sim}{P}$ of points of the shell, Fig. 2.1 assume the form

$$\underset{\sim}{P}(X^K) = \underset{\sim}{R}(X^{\Delta}) + X^3 \underset{\sim}{A}_3(X^{\Delta}) \; . \tag{2.12}$$

Thus, for the space of normal coordinates defined by (2.2), the base vectors and the components of the metric tensors are, respectively

$$\underset{\sim}{G}_{\Delta} = \underset{\sim}{P}_{,\Delta} = \underset{\sim}{A}_{\Delta} + X^3 \underset{\sim}{A}_{3,\Delta} \quad , \quad \underset{\sim}{G}_3 = \underset{\sim}{P}_{,3} = \underset{\sim}{A}_3 \tag{2.13}$$

$$G_{\Delta\Gamma} = \underset{\sim}{G}_\Delta \cdot \underset{\sim}{G}_\Gamma , \quad G_{\Delta 3} = \underset{\sim}{G}_\Delta \cdot \underset{\sim}{G}_3 = 0 \quad , \quad G_{33} = \underset{\sim}{G}_3 \cdot \underset{\sim}{G}_3 = 1 \tag{2.14}$$

With the help of (2.6) and (2.8) the equ. (2.13) becomes

$$\underset{\sim}{G}_\Delta = \underset{\sim}{A}_\Delta - X^3 B^\Gamma_\Delta \underset{\sim}{A}_\Gamma = \mu^\Gamma_\Delta \underset{\sim}{A}_\Gamma \tag{2.15}$$

where

$$\mu^\Gamma_\Delta = \delta^\Gamma_\Delta - X^3 B^\Gamma_\Delta \tag{2.16}$$

is called shifter tensor.

Values of any vector $\underset{\sim}{V}$ can be represented by their components either in the basis $\underset{\sim}{G}_\Delta$, $\underset{\sim}{G}_3$ or in the basis $\underset{\sim}{A}_\Delta$, $\underset{\sim}{A}_3$ as follows

$$\begin{aligned} \underset{\sim}{V} &= V^\Delta \underset{\sim}{G}_\Delta + V^3 \underset{\sim}{G}_3 = \bar{V}^\Delta \underset{\sim}{A}_\Delta + V^3 \underset{\sim}{A}_3 = \\ &= V_\Delta \underset{\sim}{G}^\Delta + V_3 \underset{\sim}{G}^3 = \bar{V}_\Delta \underset{\sim}{A}^\Delta + V_3 \underset{\sim}{A}^3 \end{aligned} \tag{2.17}$$

where barred quantities $\bar{V}^\Delta$, $\bar{V}_\Delta$ denote surface representation of $\underset{\sim}{V}$ and are the shifted components of this tensor.

It follows from (2.15) that the space and the surface components of the same vector are related to each other in the following manner

$$V_\Delta = \mu^\Gamma_\Delta \bar{V}_\Gamma \quad , \quad V^\Delta = (\mu^{-1})^\Delta_\Gamma \bar{V}^\Gamma \tag{2.18}$$

where $(u^{-1})^\Delta_\Gamma$ is defined by the relation

$$(u^{-1})^\Delta_\Gamma = \delta^\Delta_\Gamma + X^3 \delta^\Delta_\phi B^\phi_\Gamma + (X^3)^2 \delta^\Delta_\phi B^\phi_\theta B^\theta_\Gamma + \ldots , \quad \mu^\Gamma_\phi (\mu^{-1})^\Delta_\Gamma = \delta^\Delta_\phi \quad . \tag{2.19}$$

It should be remembered that the Christoffel symbols of the first and the second kind defined by

$$\Pi_{NKL} \equiv \frac{1}{2} (G_{NK,L} + G_{LN,K} - G_{KL,N}) \quad , \quad \Pi^M_{KL} \equiv G^{MN} \Pi_{NKL} \tag{2.20}$$

may, in view of relation $(2.14)_1$ and $\underset{\sim}{G}_{K,L} = \underset{\sim}{G}_{L,K}$, be also expressed as

$$\Pi_{NKL} = \underset{\sim}{G}_N \cdot \underset{\sim}{G}_{K,L} \quad , \quad \Pi^N_{KL} = \underset{\sim}{G}^N \cdot \underset{\sim}{G}_{K,L} \tag{2.21}$$

and hence

$$\underset{\sim}{G}_{K,L} = \Pi_{NKL}\underset{\sim}{G}^N = \Pi^N_{KL}\underset{\sim}{G}_N \quad . \tag{2.22}$$

The covariant spatial differentiation of a vector $\underset{\sim}{U}$ can be expressed as follows:

$$\begin{aligned}\underset{\sim}{U}_{,L} &= (U^K\underset{\sim}{G}_K)_{,L} = U^K_{,L}\underset{\sim}{G}_K + U^K\underset{\sim}{G}_{K,L} = \\ &= U^K_{,L}\underset{\sim}{G}_K + U^K\Pi^N_{KL}\underset{\sim}{G}_N = (U^K_{,L} + U^N\Pi^K_{NL})\underset{\sim}{G}_K = \\ &= U^K{}_{||L}\underset{\sim}{G}_K\end{aligned} \tag{2.23}$$

where by double strokes is denoted the *covariant spatial differentiation*

$$U^K{}_{||L} \equiv U^K_{,L} + U^N\Pi^K_{NL} \tag{2.24}$$

Similarly

$$\underset{\sim}{U}_{,L} = U_{K||L}\underset{\sim}{G}^K \tag{2.25}$$

where

$$U_{K||L} = U_{K,L} - U_N\Pi^N_{KL} \quad . \tag{2.26}$$

For the undeformed middle surface the Christoffel symbols will be denoted by Γ_{NKL}, Γ^N_{KL}

$$\Gamma_{NKL} \equiv \Pi_{NKL}\Big|_{X^3 = 0} \quad , \quad \Gamma^N_{KL} \equiv \Pi^N_{KL}\Big|_{X^3 = 0} \quad . \tag{2.27}$$

For a normal coordinate system $G_{\Delta 3} = 0$, $G_{33} = 1$ and in view of $(2.15)_1$, (2.20) the Christoffel symbols with two or three indices 3 vanish identically whereas the symbols with a single index 3 may be expressed in terms of the second fundamental tensor

$$\Gamma^{3}_{\Delta\Gamma} = \Gamma_{3\Delta\Gamma} = -\Gamma_{\Delta\Gamma 3} = -\Gamma_{\Delta 3\Gamma} = B_{\Delta\Gamma} \quad , \tag{2.28}$$

$$\Gamma^{\Delta}_{3\Gamma} = \Gamma^{\Delta}_{\Gamma 3} = -B^{\Delta}_{\Gamma} \quad . \tag{2.29}$$

The *covariant surface derivative* (with respect to the surface metric) will be denoted by a stroke ($|$) and when applied to a surface representation of such tensors as $\bar{U}^{\Delta}$, $\bar{U}_{\Delta}$, $\bar{T}^{\Delta}_{\Gamma}$, reads

$$\bar{U}^{K}_{|L} = \bar{U}^{K}_{,L} + \Gamma^{K}_{\phi L}\bar{U}^{\phi} \quad , \tag{2.30}$$

$$\bar{U}_{K|L} = \bar{U}_{K,L} - \Gamma^{\phi}_{KL}\bar{U}_{\phi} \quad , \tag{2.31}$$

$$\bar{T}^{K}_{L|M} = \bar{T}^{K}_{L,M} + \Gamma^{K}_{\phi M}\bar{T}^{\phi}_{L} - \Gamma^{\phi}_{LM}\bar{T}^{K}_{\phi} \quad .$$

Finally, for further use, the rules connecting the covariant *spatial* derivatives of space vectors and the covariant *surface* derivatives of the surface representation of these vectors are given as follows:

$$U^{\Delta}_{||\Gamma} = (\mu^{-1})^{\Delta}_{\phi}(\bar{U}^{\phi}_{|\Gamma} - B^{\phi}_{\Gamma}U^{3}) \quad , \tag{2.33}$$

$$U^{\Delta}_{||3} = (\mu^{-1})^{\Delta}_{\phi}\bar{U}^{\phi}_{,3} \quad , \tag{2.34}$$

$$U^{3}_{||\Delta} = U^{3}_{|\Delta} + B_{\Delta\Gamma}\bar{U}^{\Gamma} \tag{2.35}$$

$$U_{\Delta||\Gamma} = \mu^{\phi}_{\Delta}(\bar{U}_{\phi|\Gamma} - B_{\phi\Gamma}U_{3}) \quad , \tag{2.36}$$

$$U_{\Delta||3} = \mu^{\phi}_{\Delta}\bar{U}_{\phi,3} \quad , \tag{2.37}$$

$$U_{3||\Delta} = U_{3|\Delta} + B^{\phi}_{\Delta}\bar{U}_{\phi} \tag{2.38}$$

$$U^{3}_{||3} = U_{3||3} = U^{3}_{|3} = U_{3|3} = U^{3}_{,3} = U_{3,3} \quad . \tag{2.39}$$

3. ASSUMPTIONS

In this chapter we shall discuss the foundations of the geometrically non-linear theory of thin elastic-plastic shells on the basis of a modified version of the Kircchoff-Love hypothesis.

The basic kinematic assumption of the classical Kirchhoff-Love theory of shells is that the material fibres which are rectilinear and orthogonal to the undeformed middle surface S remain, after an arbitrary deformation, rectilinear and orthogonal to the deformed middle surface s, and do not change their length.

This kinematical assumption of the K-L theory is in contradiction with the statical assumption of the plane stress state for an elastic shell, and with the incompressibility condition for plastic materials.

For the theory of small strains and small displacements, the inconsistency of the K-L theory leads to errors which are found to be negligibly small. The K-L approximation becomes, however, debatable if large elastic-plastic deformations are considered.

A great number of existing analytical solutions in the geometrically non-linear plastic shell theory refer to a narrow class of shallow shells undergoing moderately large deflections and infinitesimal tangential displacements. The geometrical non-linearity enters into the strain-displacement relations in the form of an additional term involving the square of a gradient of deflection vector. The geometrical relations of this type have first been derived by Karman [1] for thin plates and then generalized to cover shallow shells by Donnell [2], Mushtari [3] and Vlasov [4].

When applying Donnell-Mushtari-Vlasov theory, the restrictions imposed on the initial shell curvature and on the magnitude of the displacements result in the elimination of numerous interesting shell structures from the considerations.

The purpose of the paper is to present the plastic shell theory which avoids those restrictions, that is the theory which allows for

- the extension of the class of the considered shells to cover non-shallow shells,

- the extension of the class of the considered deformations to account for large displacements and large strains,
- the reformulation of the K-L assumptions in order to eliminate certain inconsistencies which can no longer be accepted if large plastic deformations are considered.

The present theory will be developped within the framework of the following set of simpifying assumptions:

(i) The shell is thin, i. e. the ratio of the thickness to the smallest radius of curvature is negligible as compared with unity,

$$\frac{2H}{R_{min.}} << 1 \quad , \tag{3.1}$$

(ii) The components of the displacement vector U_K are analytic functions of X^3 - the coordinate normal to the shell middle surface. Then they may be written in the form

$$U_\Gamma = V_\Gamma + X^3\beta_\Gamma + (X^3)^2\gamma_\Gamma + \ldots \tag{3.2}$$

$$U_3 = W + X^3\beta_3 + (X^3)^2\gamma_3 + \ldots \tag{3.3}$$

where

V_Γ , W - denote the displacements in the tangential and the normal directions, respectively, of a point on the middle surface,

β_Γ , γ_Γ - stand for the inclination of the normal to the middle surface.

β_3 , γ_3 - describe the normal strain distribution.

(iii) Transverse shearing strains are negligibly small,

$$E_{\Delta 3} \simeq 0 \quad \text{for every } X^3. \tag{3.4}$$

(iv) No volume changes take place during the plastic deformation. This requirement is known as the incompressbility condition and after linearization can be written in the form

$$E_{33} = -E_{\Delta\Gamma}G^{\Delta\Gamma} \quad . \tag{3.5}$$

The influence of higher order terms may be proved to be small in the considered theory.

(v) The elastic strains are small.

One of the difficulties in any theory of shells, especially in the presence of finite strains, lies in the derivation of the strain-displacement relations. Since a shell is defined as a three-dimensional body with one of the dimensions being much smaller than the others, the deformation of a shell may be either described within the framework of surface theory, or derived from the general case of a 3-dimensional body. The latter approach will be applied in what follows.

Another problem to be faced when formulating the finite deformation shell theory is the suitable choice of description. The most commonly used in the shell theory are the total Lagrangian and the convected (sometimes called the Eulerian) descriptions.

In the total Lagrangian description all tensors and functions are referred to the initial frame of reference whereas in the convected (Eulerian) description to the reference system deforming together with the shell.

In the presented theory the total Lagrangian description will be applied, i. e. all quantities will be referred to the undeformed configuration of the shell. The stress state is thus described by the symmetric Piola-Kirchhoff stress tensor $\underset{\sim}{S}$ whereas the strain is specified by the Green strain tensor $\underset{\sim}{E}$. They constitute a conjugate pair of stress and strain measures since, as defined in [5], the rate of deformation energy per unit initial volume is given by the scalar product of these stress and strain rate tensors

$$\dot{d} = \underset{\sim}{S} \cdot \dot{\underset{\sim}{E}} \tag{3.6}$$

4. KINEMATICS

The spatial components of the Green strain tensor E_{KL} can be expressed in the terms of the spatial components of the displacement vector U_k in the following manner

$$2E_{KL} = U_{K||L} + U_{L||K} + U_{M||K} U^{M}_{||L} \quad . \tag{4.1}$$

This relation can be rewritten to become

$$2E_{\Delta\Gamma} = U_{\Delta||\Gamma} + U_{\Gamma||\Delta} + U_{\phi||\Delta} U^{\phi}_{||\Gamma} + U_{3||\Delta} U^{3}_{||\Gamma} \ , \tag{4.2}$$

$$2E_{\Delta 3} = U_{3||\Delta} + U_{\Delta||3} + U_{\phi||3} U^{\phi}_{||\Delta} + U_{3||3} U^{3}_{||\Delta} \ , \tag{4.3}$$

$$2E_{33} = 2U_{3||3} + U_{\phi||3} U^{\phi}_{||3} + U_{3||3} U^{3}_{||3} \quad . \tag{4.4}$$

With the aid of (2.33) - (2.39) the Green strain tensor (4.2) - (4.4) takes the form

$$\begin{aligned} 2E_{\Delta\Gamma} = {} & \mu^{\phi}_{\Delta}(\bar{U}_{\phi|\Gamma} - B_{\phi\Gamma}U_3) + \mu^{\phi}_{\Gamma}(\bar{U}_{\phi|\Delta} - B_{\phi\Delta}U_3) + \\ & + (\bar{U}^{\phi}_{|\Delta} - B^{\phi}_{\Delta}U_3)(\bar{U}_{\phi|\Gamma} - B_{\phi\Gamma}U_3) + \\ & + (U_{3|\Delta} + B^{\phi}_{\Delta}\bar{U}_{\phi})(U_{3|\Gamma} + B^{\theta}_{\Gamma}\bar{U}_{\theta}) \quad , \end{aligned} \tag{4.5}$$

$$\begin{aligned} 2E_{\Delta 3} = {} & \mu^{\phi}_{\Delta}\bar{U}_{\phi,3} + U_{3|\Delta} + B^{\phi}_{\Delta}\bar{U}_{\phi} + (\bar{U}^{\phi}_{|\Delta} - B^{\phi}_{\Delta}U_3)\bar{U}_{\phi,3} + \\ & + (U_{3|\Delta} + B_{\phi\Delta}\bar{U}^{\phi})U_{3,3} \quad , \end{aligned} \tag{4.6}$$

$$2E_{33} = 2U_{3,3} + \bar{U}^{\Delta}_{,3}\bar{U}_{\Delta,3} + (U_{3,3})^2 \quad . \tag{4.7}$$

Let us restrict further considerations to thin shells. In view of assumption (i)

$$\mu^{\Delta}_{\Gamma} \simeq \delta^{\Delta}_{\Gamma} \tag{4.8}$$

and

$$\bar{U}_\Delta \simeq U_\Delta \ . \tag{4.9}$$

Substituting (4.8), (4.9) (3.2), (3.3) into (4.5) - (4.7) and omitting nonlinear terms with respect to X^3 (which, again, in view of the assumption (i) are negligibly small), we obtain

$$2E_{\Delta\Gamma} = \Lambda_{\Delta\Gamma} + X^3 K_{\Delta\Gamma} \ , \tag{4.10}$$

$$2E_{\Delta 3} = \Lambda_{\Delta 3} + X^3 K_{\Delta 3} \ , \tag{4.11}$$

$$2E_{33} = \Lambda_{33} + X^3 K_{33} \ , \tag{4.12}$$

where

$$\Lambda_{\Delta\Gamma} = V_{(\Delta|\Gamma)} - B_{\Delta\Gamma} W + \frac{1}{2} (W|_\Delta W|_\Gamma + B^\phi_\Delta B_{\phi\Gamma} W^2 + + V^\phi|_{(\Delta} V_{\phi|\Gamma)} + B^\phi_\Delta B^\theta_\Gamma V_\theta V_\phi) + B^\phi_{(\Delta} V_\phi W|_{\Gamma)} - B^\phi_{(\Delta} V_{\phi|\Gamma)} W \tag{4.13}$$

$$\Lambda_{\Delta 3} = \frac{1}{2} (\beta_\Delta + W|_\Delta + B^\phi_\Delta V_\phi + V^\phi|_\Delta \beta_\phi + B^\phi_\Delta \beta_\phi W + + W|_\Delta \beta_3 + B^\phi_\Delta V_\phi \beta_3) \ , \tag{4.14}$$

$$\Lambda_{33} = \beta_3 + \frac{1}{2} (\beta^\Delta \beta_\Delta + \beta^2_3) \ , \tag{4.15}$$

$$K_{\Delta\Gamma} = \beta_{(\Delta|\Gamma)} - B_{\Delta\Gamma}\beta_3 + V^\phi|_{(\Delta}\beta_{\phi|\Gamma)} - B^\phi_{(\Delta}\beta_{\phi|\Gamma)} W + + B^\phi_{(\Delta} B_{\phi\Gamma)} W\beta_3 - B^\phi_{(\Delta} V_{\phi|\Gamma)}\beta_3 + W|_{(\Delta}\beta_{3|\Gamma)} + B^\phi_{(\Delta} V_\phi \beta_{3|\Gamma)} + + B^\phi_{(\Gamma} W|_{\Delta)}\beta_\phi + B^\phi_{(\Delta} B^\theta_{\Gamma)} V_\theta \beta_\phi \ . \tag{4.16}$$

$$K_{\Delta 3} = \frac{1}{2}(\beta_{3|\Delta} + B^\phi_\Delta \beta_\phi + \beta^\phi|_\Delta \beta_\phi - B^\phi_\Delta \beta_\phi \beta_3 + \beta_{3|\Delta}\beta_3 + + B^\phi_\Delta \beta_\phi \beta_3) + \gamma_\phi V^\phi|_\Delta + \gamma_\Delta - 2B^\phi_\Delta \gamma_\phi W + \gamma_3 W|_\Delta + B^\Gamma_\Delta \gamma_3 V_\Gamma , \tag{4.17}$$

$$K_{33} = 2\gamma_3 - B^{\Gamma}_{\phi}\beta_{\Gamma}\beta^{\phi} + 2\gamma_{\phi}\beta^{\phi} + 2\gamma_3\beta_3 \quad . \tag{4.18}$$

The bracketed indices denote symmetrization.

In order to describe the strain state solely by means of the displacement components of the middle surface, the functions β_{Δ}, β_3, γ_{Δ} and γ_3 should be expressed in terms of W and V . To this end we make use of the assumptions (iii) and (iv).

The condition (3.4) of vanishing transverse strains yields two equations:

$$\Lambda_{\Delta 3} = 0 \tag{4.19}$$

$$K_{\Delta 3} = 0 \tag{4.20}$$

whereas the incompressibility condition (3.5) to be satisfied for an arbitrary X^3 gives two more requirements

$$\Lambda_{33} = -\Lambda_{\Delta\Gamma}G^{\Delta\Gamma} \tag{4.21}$$

$$K_{33} = -K_{\Delta\Gamma}G^{\Delta\Gamma} \tag{4.22}$$

if nonlinear terms with respect to X^3 are neglected.

The equations (4.19) - (4.22) constitute the set of four equations in four unknown quantities β_{Δ}, β_3, γ_{Δ}, γ_3.

It may be shown, [6], that the influence of γ_{Δ} and γ_3 terms on the dissipation of energy is of the order of magnitude of $(2H/R)^2$ or $(2H/L)^2$ as compared with unity and thus, in view of the assumption (i), is negligible.

5. CLASSIFICATION AND SIMPLIFICATIONS

The general relations (4.13) - (4.18) for the extension and for the change of curvature of the middle surface of a shell are rather involved. They become even more complicated if all quantities are written explicitly in terms of the displacement components and their derivatives.

Numerous attempts have been made to simplify and classify the basic relations for shells subject to large deformations. Important contributions in the field are due to Chien [6], Donnell [7], Reissner [8], Koiter [9], John [10], Naghdi [11], Sanders [12, 13], Danielson [14], Simmonds and Danielson [15], and Pietraszkiewicz [16]. These papers are concerned, however, with the analysis of elastic shells and the proposed geometric relations are not fully adequate for the description of plastic shells. The need thus arises to formulate geometrically non-linear shell theory to satisfy the requirements and specifications of plasticity.

To fulfil this need, we shall present a number of approximated geometrically non-linear theories formulated on the basis of the assumptions made in chapter 4.

To obtain a consistently simplified theory, all approximations used in the basic relations have to be made to with in the same degree of accuracy. To this end, we first establish the order of magnitude of each individual term in the geometrical relations and next, we omit all terms which are smaller than, or equal to a small number ε^2 as compared with the largest term. This small number ε^2 is chosen as the order of magnitude of the ratio of shell thickness to the smallest radius of curvature,

$$\varepsilon^2 \equiv O\left(\frac{2H}{R_{min.}} \right) \quad . \tag{5.1}$$

In the shell theory we usually assume that $\varepsilon = 10^{-1}$.

If the external surface loads do not vary rapidly over the middle surface, it may be assumed that the deformation wave length 2L is of the same order of magnitude as characteristic dimension of a shell (for instance the length in the case of a cylindrical shell or the diameter of a cap shell, Fig. 5.1).

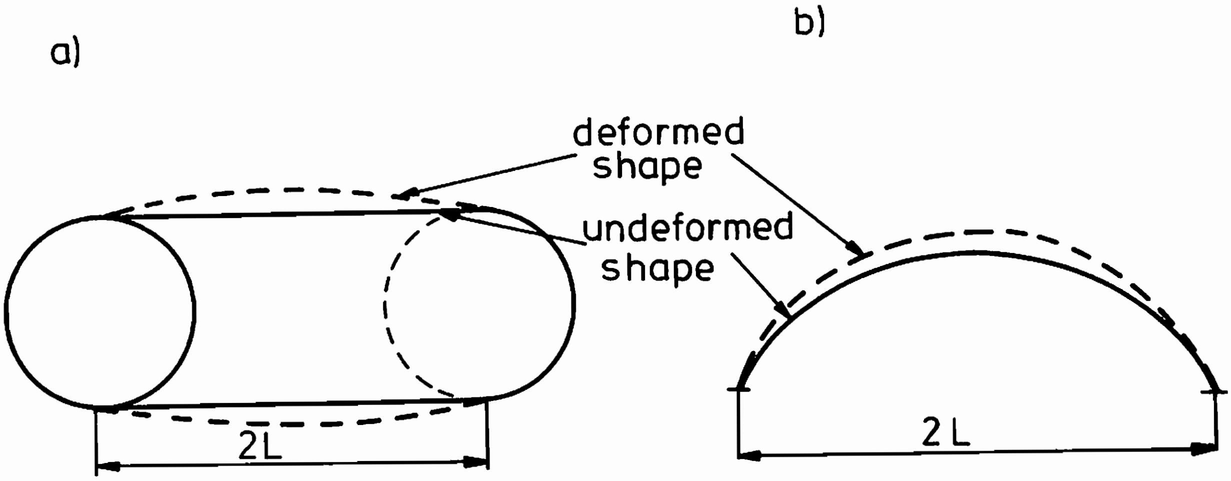

Fig. 5.1

The above assumptions enable us to estimate the order of magnitude of the covariant derivatives of the displacement components, namely

$$|w_{|\Gamma}| \le O\left(\frac{W}{L}\right) \quad , \quad |v_{\Delta|\Gamma}| \le O\left(\frac{V_\Delta}{L}\right) \quad . \tag{5.2}$$

Therefore, in order to select appropriately all the terms in the geometrical relations (4.13) - (4.18) (and subsequently in the equilibrium equations), we should introduce parameters characterizing not only the plastic deformations but also the initial shape of the shell. The following classification of the shapes of shells with respect to their shallowness is proposed.

1. Shallow shells

$$L/R_{min.} \le O(\varepsilon) \quad , \quad \text{or} \quad L \lesssim 0{,}1\ R_{min.}$$

2. Quasi-shallow shells

$$L/R_{min.} = O(1) \quad , \quad \text{or} \quad L \approx R_{min.}$$

3. Deep shells

$$(L/R_{min})^{-1} \geq O(\varepsilon) \quad , \quad \text{or} \quad L \gtrsim 10\, R_{min}.$$

Now, we attempt to classify the deformations. Since, in view of the assumptions i - iv, the deformation of a shell is determined by the normal W and the tangential V_Δ components of the displacement vector of the middle surface, it is useful to introduce the following classification of the deformation mode.

a) Small W and small V_Δ

$$\max.\left(\frac{W}{2H}\right) \leq O(\varepsilon) \quad , \quad \max.\left(\frac{V_\Delta}{2H}\right) \leq O(\varepsilon) \ .$$

b) Moderately large W and small V_Δ

$$\max.\left(\frac{W}{2H}\right) = O(1), \quad \max.\left(\frac{V_\Delta}{2H}\right) \leq O(\varepsilon) \ .$$

c) Large W and small V_Δ

$$\max.\left(\frac{W}{2H}\right) = O(\varepsilon^{-1}), \max.\left(\frac{V_\Delta}{2H}\right) \leq O(\varepsilon) \ .$$

The more detailed classification of the initial shape and the deformation mode of a shell is presented in [50].

Once the initial shape of a considered shell is given and the deformation mode is predicted or assumed, the order of magnitude of each term in the expression for extension $\Lambda_{\Delta\Gamma}$ can be established.

Next, neglecting those terms which for the given theory are considered to be small (smaller than, or equal to, ε^2 as compared with the largest term), we obtain, for different degrees of shallowness (deepness) and different deformation modes, various sets of geometrical relations, as listed in Table 5.1.

As a result we obtain an effective method of simplifying the geometrical relations. There can arise, however, a situation which must not be overlooked. The greates terms (of the lowest order of magnitude) with which the remaining terms are compared, can cancel mutually so that the neglected terms are not in fact small as compared with the algebraic sum of the remaining terms. Having this in view, we retain in the relations for $\Lambda_{\Delta\Gamma}$ also all the linear terms.

On the derivation of the geometrically non-linear theory for elastic shells, the linearized formula for the change of curvature has been generally agreed. When defining a measure of the change of curvature for plastic shells, we have to consider a contribution of bending terms in the expression for the energy dissipation during the plastic flow. For a stable deformation process in a shell there usually takes place a gradual transition from a flexural to a membrane state, so that the contribution of bending in the energy dissipation diminishes as the deformation process goes on (see e.g. [25] - [27]).Therefore, in a nonlinear formulation, we complement a linear expression for the change of curvature by such nonlinear terms which are of the same order of magnitude as the greatest linear term and, moreover, we neglect linear terms which are sufficiently small.

When considering buckling problems as well as the postbuckling behaviour, the relations for the change of curvature should be constructed even more carefully.

We shall present now the simplified geometrical relations, listed in the Table 5.1., for various geometrically non-linear shell theories classified with respect to their initial shape and deformation mode.

The first column identifies the respective theory according to the assumed classification. The first symbol in the column 1 refers to the type of inital shell geometry specified in columns 2,3,4 by parameters 2H/R and L/R. The second symbol refers to the type of deformation mode, specified in columns 5,6 and 7 by parameters W/2H and V_Δ/2H. The last two columns provide expressions for the approximate kinematic relations, appropriate for the given theory.

The cases 1a, 2a, 3a describe the geometrically linear theory of infinitesimal strains and displacements. The expressions for the strain tensor $\Lambda_{\Delta\Gamma}$ are then identical whereas the change of curvature $K_{\Delta\Gamma}$ for

cases	Type of shell	2H/R	L/R	Type of deformation	W/2H	$V_\Delta/2H$	$\Lambda_{\Delta\Gamma}$	$K_{\Delta\Gamma}$
1	2	3	4	5	6	7	8	9
1a	thin, shallow	ε^2	ε	small W, small V_Δ	ε	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W$	$-W_{\|\Delta\Gamma} - B^\phi_{\Delta\|\Gamma}V_\phi - B^\phi_{(\Delta}V_{\phi\|\Gamma)} + B_{\Delta\Gamma}V^\phi_{\|\phi}$
1b				moderately large W, small V_Δ	1	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W + \frac{1}{2} W_{\|\Delta}W_{\|\Gamma}$	$-W_{\|\Delta\Gamma}$
1c				large W, small V_Δ	ε^{-1}	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W + \frac{1}{2} W_{\|\Delta}W_{\|\Gamma}$	$-W_{\|\Delta\Gamma} + W_{\|\Delta\Gamma}W_{\|\phi}W_{\|}^{\phi}$
2a	thin, quasi-shallow	ε^2	1	small W, small V_Δ	ε	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W$	$-W_{\|\Delta\Gamma} - B^\phi_{\Delta\|\Gamma}V_\phi - B^\phi_{(\Delta}V_{\phi\|\Gamma)} + B_{\Delta\Gamma}(V^\phi_{\|\phi} - B^\phi_\phi W)$
2b				moderately large W small V_Δ	1	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W$	$-W_{\|\Delta\Gamma} - B^\phi_{\Delta\|\Gamma}V_\phi - B^\phi_{(\Delta}V_{\phi\|\Gamma)} + B_{\Delta\Gamma}(V^\phi_{\|\phi} - B^\phi_\phi W)$
2c				large W, small V_Δ	ε^{-1}	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W + \frac{1}{2} W_{\|\Delta}W_{\|\Gamma} + B^\phi_\Delta B_{\phi\Gamma}W^2$	$-W_{\|\Delta\Gamma} - B_{\Delta\Gamma}B^\phi_\phi W$
3a	thin, deep	ε^2	ε^{-1}	small W, small V_Δ	ε	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W$	$-W_{\|\Delta\Gamma} - B^\phi_{\Delta\|\Gamma}V_\phi - B^\phi_{(\Delta}V_{\phi\|\Gamma)} + B_{\Delta\Gamma}(V^\phi_{\|\phi} - B^\phi_\phi W)$
3b				moderately large W small V_Δ	1	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W$	$-W_{\|\Delta\Gamma} - B_{\Delta\Gamma}B^\phi_\phi W$
3c				large W, small V_Δ	ε^{-1}	ε	$V_{(\Delta\|\Gamma)} - B_{\Delta\Gamma}W + \frac{1}{2} B^\phi_\Delta B_{\phi\Gamma}W^2$	$-W_{\|\Delta\Gamma} - B_{\Delta\Gamma}B^\phi_\phi W$

Table 5.1

shallow shells can be simplified by neglecting the term $B_{\Delta\Gamma}B^{\phi}_{\phi}W$. This term causes that, when considering e. q. cylindrical shells, we have to take into account the changes of circumferential curvature even for rotationally symmetric deformation.

It is seen that the cases 1a and 1b for shallow shells, 2a and 2b for quasi-shallow shells and 3a, 3b for deep shells all correspond to small strains (max $\Lambda_{\Delta\Gamma} \leq \varepsilon^2 << 1$). In the remaining cases the middle surface extension tensor remains no longer small (max $\Lambda_{\Delta\Gamma} > \varepsilon^2$).

The relations obtained in the case 1b, i.e. for shallow shells subject to moderately large deflections and small tangential displacements, were extensively studied in the literature and are referred to as the Donnel-Vlasov equations.

This case is interesting in that the strains are infinitesimal and yet the strain displacement relations involve a nonlinear terms.

However, for the quasi-shallow and deep shells at the moderately large deflections (cases 2b and 3b in the Table 5.1), the geometrically non-linear terms in the relations for the strain tensor $\Lambda_{\Delta\Gamma}$ are of the order of magnitude ε^2 as compared with the term $B_{\Delta\Gamma}W$ and can thus be neglected. Therefore the geometrically linear theory is in these cases sufficiently accurate.

For large deflections (the cases 1c, 2c, 3c in the Table 5.1), the strain tensors $\Lambda_{\Delta\Gamma}$ comprise different non-linear terms for the shells characterized by different shallowness parameter L/R.

For the shallow shells at large displacements the expression describing the change of curvature $K_{\Delta\Gamma}$ involves the non-linear term $W_{|\Delta\Gamma}W_{|\phi}W_{|}^{\phi}$ which is of the same order of magnitude as the linear term $W_{|\Delta\Gamma}$.

6. STRESS RESULTANTS

In the analysis of thin structures such as shells or plates it is convenient to deal with stress resultants rather than with the stresses themselves.

In order to get a consistent set of stress resultants and strain rates the virtual energy principle will be used. The rate of strain energy per unit reference volume, specified in terms of the Lagrangian variables, is given by the expression (3.6):

$$\dot{d} = S^{KL}\dot{E}_{KL} = S^{\Delta\Gamma}\dot{E}_{\Delta\Gamma} + S^{\Delta 3}\dot{E}_{\Delta 3} + S^{3\Gamma}\dot{E}_{3\Gamma} + S^{33}\dot{E}_{33} \ . \tag{6.1}$$

Making use of the assumptions (iii) and (iv), chapter (3), i. e. equations (3.4) and (3.5), the equation (6.1) can be rewritten in the form:

$$\dot{d} = S^{\Delta\Gamma}\dot{E}_{\Delta\Gamma} - S^{33}\dot{E}_{\Delta\Gamma}G^{\Delta\Gamma} = (S^{\Delta\Gamma} - S^{33}G^{\Delta\Gamma})\dot{E}_{\Delta\Gamma} \tag{6.2}$$

or

$$\dot{d} = \hat{S}^{\Delta\Gamma}\dot{E}_{\Delta\Gamma} \tag{6.3}$$

where

$$\hat{S}^{\Delta\Gamma} = S^{\Delta\Gamma} - S^{33}G^{\Delta\Gamma} \ . \tag{6.4}$$

Therefore the stress measure $\hat{S}^{\Delta\Gamma}$ can be treated as conjugate with the strain measure $E_{\Delta\Gamma}$.

If we define the strain energy per unit area of the middle surface by

$$\dot{\bar{d}} = \int_{-H}^{+H} \mu\dot{d}\, dX^3 \tag{6.5}$$

where $\mu = \det \mu^{\Delta}_{\Gamma}$ then, under the assumption (4.8), $\mu = 1$ and in view of (6.3) we may write

$$\dot{\bar{d}} = \int_{-H}^{+H} \hat{S}^{\Delta\Gamma}\dot{E}_{\Delta\Gamma}\, dX^3 \ . \tag{6.6}$$

Substituting (4.10) into (6.6), we obtain

$$\dot{\bar{d}} = \dot{\Lambda}_{\Delta\Gamma} \int_{-H}^{+H} \hat{S}^{\Delta\Gamma}\, dx^3 + \dot{K}_{\Delta\Gamma} \int_{-H}^{+H} x^3 \hat{S}^{\Delta\Gamma}\, dx^3 \ . \tag{6.7}$$

Next, introducing stress resultants and stress couples by the definitions

$$N^{\Delta\Gamma} = \int_{-H}^{+H} \hat{S}^{\Delta\Gamma} dx^3 \quad , \tag{6.8}$$

$$M^{\Delta\Gamma} = \int_{-H}^{+H} x^3 \hat{S}^{\Delta\Gamma} dx^3 \ , \tag{6.9}$$

the equation (6.7) becomes

$$\dot{\bar{d}} = N^{\Delta\Gamma}\dot{\Lambda}_{\Delta\Gamma} + M^{\Delta\Gamma}\dot{K}_{\Delta\Gamma} \ . \tag{6.10}$$

The relation (6.10) indicates that $N^{\Delta\Gamma}$ and $M^{\Delta\Gamma}$ as defined by equations (6.8) and (6.9) are the generalized stress measures conjugate with the generalized strain measures $\Lambda_{\Delta\Gamma}$, $K_{\Delta\Gamma}$.

In most of the examples to follow, it is convenient to deal with dimensionless quantities. We denote by N_o the maximum direct stress which the shell can withstand in uniaxial tension, by M_o the maximum pure bending moment, and define the dimensionless stress resultants as follows:

$$n^{\Delta\Gamma} \equiv \frac{N^{\Delta\Gamma}}{N_o} \quad , \qquad m^{\Delta\Gamma} = \frac{M^{\Delta\Gamma}}{M_o} \ . \tag{6.11}$$

For a shell of uniform thickness 2H and the tensile yield stress σ_o we have

$$N_o = 2H\sigma_o \quad , \qquad M_o = \sigma_o H^2 \ . \tag{6.12}$$

The conjugate dimensionless kinematical variables, viz., the extension and curvature reates, are

$$\dot{\lambda}_{\Delta\Gamma} \equiv \dot{\Lambda}_{\Delta\Gamma} \qquad \dot{\kappa}_{\Delta\Gamma} \equiv \frac{M_o}{N_o}\dot{K}_{\Delta\Gamma} \tag{6.13}$$

so that the dissipation density per unit area of the reference surface is

$$\dot{\bar{d}} = (n^{\Delta\Gamma}\dot{\lambda}_{\Delta\Gamma} + m^{\Delta\Gamma}\dot{\kappa}_{\Delta\Gamma})N_o \ . \tag{6.14}$$

7. EQUILIBRIUM EQUATIONS

There are two approaches to the derivation of two-dimensional equilibrium conditions for shells in the Lagrangian description.

In the first one we start with the relations describing equilibrium of a material point in a three-dimensional spatial position as referred to the undeformed configuration. Next, we refer all the quantities to a base on the middle surface, integrate across the shell thickness and introduce the stress and couple resultants (per unit length of curves on the middle surface).

The other approach consists in derivation of the equilibrium equations and the natural boundary conditions from the *two-dimensional virtual work principle*. For any exact theory both approaches lead obviously to the same results. For approxmate theories the simplifications made when deriving particular groups of relations can cause certain differences.

In the present problem, we insist on making the approximations in such a way that the principle of virtual work, as a fundamental law of continuum mechanics, be satisfied completely. To this end the second approach to the derivation of equilibrium conditions will be employed.

The principle of the rate of virtual work reads

$$\dot{D}_{ext} = \dot{D}_{int} \tag{7.1}$$

where $\dot{D}_{ext}$ denotes the rate of virtual work of the external forces (loading and boundary forces), whereas $\dot{D}_{int}$ stands for the rate of the work done by internal forces.

We consider a shell under external loads P applied to its middle surface

$$\underset{\sim}{P} = P^{\Delta}\underset{\sim}{A}_{\Delta} + P^{3}\underset{\sim}{A}_{3} \tag{7.2}$$

and under external forces $\bar{\underset{\sim}{N}}$, $\bar{\underset{\sim}{Q}}$, $\bar{\underset{\sim}{M}}$ applied on the boundary C with the normal $\underset{\sim}{N}$.

The rate of external virtual energy is then given by the integrals taken over the middle surface S and the boundary curve C:

$$\dot{D}_{ext.} = \int_S (P^\Delta \dot{V}_\Delta + P^3 \dot{W})\, dS +$$

$$+ \int_C (\bar{N}^{\Delta\Gamma} \dot{V}_\Delta + \bar{Q}^\Gamma \dot{W} + \bar{M}^{\Delta\Gamma} \dot{\beta}_\Delta) N_\Gamma \, dC \tag{7.3}$$

where β_Δ is the rotation vector at the boundary and can be calculated by means of the equation (4.19).

The rate of internal virtual energy is obtained on integrating (6.10) over the middle surface S.

$$\dot{D}_{int.} = \int_S (N^{\Delta\Gamma} \dot{\Lambda}_{\Delta\Gamma} + M^{\Delta\Gamma} \dot{K}_{\Delta\Gamma})\, dS \quad . \tag{7.4}$$

As examples we shall derive the equilibrium equations for the shells defined in Table 5.1 in the cases 1.b and 2.c.

<u>Case 1.b. - shallow shell at moderately large W and small V_Δ</u>

According to the Table 5.1, the strain rate and the curvature rate tensors of the middle surface are

$$\dot{\Lambda}_{\Delta\Gamma} = \dot{V}_{(\Delta|\Gamma)} - B_{\Delta\Gamma} \dot{W} + \dot{W}_{|\Delta} W_{|\Gamma} \quad , \tag{7.5}$$

$$\dot{K}_{\Delta\Gamma} = -\dot{W}_{|\Delta\Gamma} \quad . \tag{7.6}$$

Substitution of (7.5) and (7.6) into (7.4) furnishes

$$\dot{D}_{int.} = \int_S [N^{\Delta\Gamma} (\dot{V}_{\Delta|\Gamma} - B_{\Delta\Gamma} \dot{W} + W_{|\Delta} \dot{W}_{|\Gamma}) - M^{\Delta\Gamma} \dot{W}_{|\Delta\Gamma}]\, dS \quad . \tag{7.7}$$

We now eliminate the derivatives of the velocities by using the Green-Gauss theorem. Hence

$$\dot{D}_{int.} = \int_S [-N^{\Delta\Gamma}{}_{|\Gamma} \dot{V}_\Delta - N^{\Delta\Gamma} B_{\Delta\Gamma} \dot{W} - (N^{\Delta\Gamma} W_{|\Delta})_{|\Gamma} \dot{W} - M^{\Delta\Gamma}{}_{|\Delta\Gamma} \dot{W}]\, dS$$

$$+ \int_C (N^{\Delta\Gamma} \dot{V}_\Delta + N^{\Delta\Gamma} W_{|\Delta} \dot{W} - M^{\Delta\Gamma} \dot{W}_{|\Delta} + M^{\Delta\Gamma}{}_{|\Delta} \dot{W}) N_\Gamma \, dC. \tag{7.8}$$

Employing the principle of virtual energy (7.1) to the rate of external and internal energy given respectively by (7.3) and (7.8) we obtain

$$\int_S \{(P^\Delta + N^{\Delta\Gamma}{}_{|\Gamma})\dot{V}_\Delta + [P^3 + N^{\Delta\Gamma}B_{\Delta\Gamma} + (N^{\Delta\Gamma}W_{|\Delta})_{|\Gamma} + M^{\Delta\Gamma}{}_{|\Delta\Gamma}]\dot{W}\}dS +$$

$$+ \int_C [(\bar{N}^{\Delta\Gamma} - N^{\Delta\Gamma})\dot{V}_\Delta + (\bar{Q}^\Gamma - N^{\Delta\Gamma}W_{|\Delta} - M^{\Delta\Gamma}{}_{|\Delta})\dot{W} + (\bar{M}^{\Delta\Gamma} - M^{\Delta\Gamma})\dot{\beta}_\Delta]N_\Gamma dC = 0. \quad (7.9)$$

If the equation (7.9) holds for all virtual velocity fields which satisfy geometrical relations (7.5), (7.6) and the displacement and slope boundary conditions, then the generalized stresses must satisfy the following relations:

$$N^{\Delta\Gamma}{}_{|\Gamma} + P^\Delta = 0 \quad , \tag{7.10}$$

$$N^{\Delta\Gamma}B_{\Delta\Gamma} + (N^{\Delta\Gamma}W_{|\Delta})_{|\Gamma} + M^{\Delta\Gamma}{}_{|\Delta\Gamma} + P^3 = 0 \tag{7.11}$$

for any point of the middle surface S and

$$\bar{N}^{\Delta\Gamma}N_\Gamma = N^{\Delta\Gamma}N_\Gamma \quad , \tag{7.12}$$

$$\bar{M}^{\Delta\Gamma}N_\Gamma = M^{\Delta\Gamma}N_\Gamma \quad , \tag{7.13}$$

$$\bar{Q}^\Gamma N_\Gamma = (N^{\Delta\Gamma}W_{|\Delta} + M^{\Delta\Gamma}{}_{|\Delta})N_\Gamma \tag{7.14}$$

for any point of the boundary curve C. Hence, the equations (7.10) - (7.11) constitute the equilibrium conditions and the equations (7.12)- (7.14) form the natural boundary conditions.

As we can see the shear forces Q^Δ do not appear in the equilibrium equations (7.10), (7.11) and may therefore be treated not as generalized stresses but as reactions. This is the consequence of neglecting the shear strains, so that shear forces take no part in the internal virtual energy.

<u>Case 2.c. - quasi-shallow shell at large W and small V_Δ</u>

According to the Table 5.1, the strain rate and the curvature rate tensors are:

$$\dot{\Lambda}_{\Delta\Gamma} = \dot{V}_{(\Delta|\Gamma)} - B_{\Delta\Gamma}\dot{W} + W_{|\Delta}\dot{W}_{|\Gamma} + B^{\phi}_{\Delta}B_{\phi\Gamma}W\dot{W} \ , \tag{7.15}$$

$$\dot{K}_{\Delta\Gamma} = -\dot{W}_{|\Delta\Gamma} - B_{\Delta\Gamma}B^{\phi}_{\phi}\dot{W}. \tag{7.16}$$

Applying the same procedure as before, i. e. substituting (7.15)-(7.16) into (7.4), using Green's theorem to eliminate velocity gradients and employing the principle of virtual energy, we obtain the equilibrium equations and the boundary conditions in the following form:

$$N^{\Delta\Gamma}_{|\Gamma} + P^{\Delta} = 0 \tag{7.17}$$

$$N^{\Delta\Gamma}B_{\Delta\Gamma} + (N^{\Delta\Gamma}W_{|\Delta})_{|\Gamma} - N^{\Delta\Gamma}B^{\phi}_{\Delta}B_{\phi\Gamma}W + M^{\Delta\Gamma}_{|\Delta\Gamma} + + M^{\Delta\Gamma}B_{\Delta\Gamma}B^{\phi}_{\phi} + P^{3} = 0 \tag{7.18}$$

$$\bar{N}^{\Delta\Gamma}N_{\Gamma} = N^{\Delta\Gamma}N_{\Gamma} \ , \tag{7.19}$$

$$\bar{M}^{\Delta\Gamma}N_{\Gamma} = M^{\Delta\Gamma}N_{\Gamma} \ , \tag{7.20}$$

$$\bar{Q}^{\Gamma}N_{\Gamma} = (N^{\Delta\Gamma}W_{|\Delta} + M^{\Delta\Gamma}_{|\Delta})N_{\Gamma} \ . \tag{7.21}$$

Comparison of equs. (7.10) - (7.14) with (7.17) - (7.21) shows that in the case 2c two additional terms appear in the equilibrium equation (7.18).

8. YIELD CONDITIONS FOR SHELLS

8.1. General remarks

As we could see from the preceding chapters the kinematics and statics of thin shells are formulated in terms of quantities which are referred to the middle surface of a shell. Therefore the constitutive equations which define the shell material by relationships between stresses and strains should be replaced by corresponding relationships between the generalized stresses (moments and membrane forces) and the deformation of the middle surface.

Neglecting the contribution of the shear forces to the yielding of the section of the shell, a yield condition can be expressed as a closed, convex hypersurface in the space of generalized stresses,

$$F(\underset{\sim}{M},\underset{\sim}{N}) = 0 \tag{8.1}$$

The shape of such a yield surface depends on the shell cross-section (e. g. sandwich, uniform) and on the yield properties of the material. A given section of the shell becomes plastic when the generalized stresses are represented by a point on the yield surface.

Yield conditions for shells can be either

i) formulated directly in terms of stress resultants and stress couples defined on the middle surface of a shell $F(\underset{\sim}{M},\underset{\sim}{N}) = 0$, or

ii) derived through appropriate transformation of a yield condition given in terms of Cauchy (true) stress tensor to the form involving the stress resultants and stress couples, [18] - [20].

The second approach seems to be preferable since the yield criterion (criterion of transition into a plastic state) has the physical sense when considered in an actual configuration in terms of the Cauchy stress tensor $\underset{\sim}{\sigma}$. Accepting this approach, the derivations given in [19], [20] will be mainly followed here.

If the material (total Lagrangian) description is employed the yield condition has to be first transformed to be expressed in terms of the second Piola-Kirchhoff stress tensor $\underset{\sim}{S}$ and then to the form involving the stress resultants and stress couples defined by the eqs. (6.8) and (6.9).

8.2. Huber-Mises yield condition for uniform shell

8.2.1. General form

M. T. Huber suggested [21] that there is a certain critical value of the shear energy in an elastic body responsible for the onset of yielding irrespective of the type of stress state. Huber's idea was independently expressed by R. von Mises [22] in a different form. He assumed that yielding of the material begins when the *stress intensity* $\sigma_i = \sqrt{J_2}$ (where J_2 is the second invariant of the deviatoric part of the stress tensor) reaches a critical value k. Thus the Huber-Mises yield condition assumes a particularly simple form, namely

$$J_2 = k^2 \tag{8.2}$$

where

$$J_2 = (\sigma_{ij} - \frac{1}{3}\sigma^k_k g_{ij})(\sigma^{ij} - \frac{1}{3}\sigma^l_l g^{ij}) \ , \tag{8.3}$$

$$k = \frac{1}{\sqrt{3}}\sigma_o \ , \tag{8.4}$$

$\underset{\sim}{g}$ is the metric tensor in the spatial reference system and σ_o denotes the yield stress in uniaxial tension.

Substituting (8.3) and (8.4) into (8.2), the Huber-Mises yield criterion can be rewritten in the form

$$3g_{im}g_{jn}\sigma^{ij}\sigma^{mn} - (\sigma^{ij}g_{ij})^2 = 2\sigma_o^2 \ . \tag{8.5}$$

Making use of the transformation rule $\sigma^{ij} = \frac{\rho_o}{\rho} S^{KL} x^i_{,K} x^j_{,L}$ (where for incompressible material we put $\rho_o = \rho$) the yield condition (8.5) can be expressed as follows

$$3g_{im}g_{jn}x^i_{,K}x^j_{,L}x^m_{,M}x^m_{,N}S^{KL}S^{MN} - (g_{ij}x^i_{,K}x^j_{,L}S^{KL})^2 = 2\sigma_o^2 \quad . \tag{8.6}$$

Since the Green strain tensor is defined as

$$2E_{KL} = g_{kl}x^k_{,K}x^l_{,L} - G_{KL} \tag{8.7}$$

the following relation can be written:

$$g_{kl}x^k_{,K}x^l_{,L} = G_{KL} + 2E_{KL} \quad . \tag{8.8}$$

Substituting (8.8) into (8.6), we obtain the yield condition in the form

$$F = 3(G_{KM} + 2E_{KM})(G_{LN} + 2E_{LN})S^{KL}S^{MN} -$$
$$[(G_{KL} + 2E_{KL})S^{KL}]^2 - 2\sigma_o^2 = 0 \quad . \tag{8.9}$$

It is seen that the yield condition, if transformed to the undeformed configuration, depends on deformation and takes quite an involved form. Imposing, however, the requirement that the maximum component of the strain tensor $\underset{\sim}{E}$ is small in comparison with unity

$$\max.\ E_{KL} \ll 1 \quad , \tag{8.10}$$

the condition (8.9) can be simplified to become

$$F = 3G_{KM}G_{LN}S^{KL}S^{MN} - (G_{KL}S^{KL})^2 - 2\sigma_o^2 = 0 \quad . \tag{8.11}$$

Therefore considering shells in the Lagrangian description at small deformations, the yield condition is usually taken in the same form as in the Eulerian description. The simplified yield condition can not be applied, however, when deriving loading criterion, because the rate of change of the yield functions before and after simplification may differ significantly even at the yield point load (for $E_{KL} = 0$).

Let us discuss this problem for the considered case of the Huber-Mises yield condition.

The loading criterion

$$F = 0 \text{ and } \dot{F} = 0 \tag{8.12}$$

for the simplified yield condition (8.11) leads to the relation

$$\dot{F} = \frac{dF}{dS^{KL}}\dot{S}^{KL} = (6G_{KM}G_{LN}S^{MN} - 2\,G_{MN}S^{MN}G_{KL})\dot{S}^{KL} = 0 \tag{8.13}$$

whereas for the exact yield condition (8.9) we have

$$\dot{F} = \frac{\partial F}{\partial S^{KL}}\dot{S}^{KL} + \frac{\partial F}{\partial E_{KL}}\dot{E}_{KL} =$$

$$= [6(G_{KM} + 2E_{KM})(G_{LN} + 2E_{LN})S^{MN} - 2(G_{MN} + 2E_{MN})S^{MN}(G_{KL} + 2E_{KL})]\dot{S}^{KL} +$$

$$+12(G_{KM} + 2E_{KM})S^{KL}S^{MN}\dot{E}_{LN} - 4(G_{MN} + 2E_{MN})S^{MN}S^{KL}\dot{E}_{KL} = 0\ . \tag{8.14}$$

The assumption (8.10) that the components of the strain tensor $\underset{\sim}{E}$ are small in comparison with unity (or even $E_{KL} = 0$) when applied to (8.14) gives:

$$\dot{F} = (6G_{KM}G_{LN}S^{MN} - 2G_{MN}G_{KL}S^{MN})\dot{S}^{KL} + 12G_{KM}S^{KL}S^{MN}\dot{E}_{LN} - 4G_{MN}S^{MN}S^{KL}\dot{E}_{KL} =$$

$$= 6(G_{LN}\dot{S}^{KL} + 2S^{KL}\dot{E}_{LN})G_{KM}S^{MN} - 2(G_{KL}\dot{S}^{KL} + 2S^{KL}\dot{E}_{KL})G_{MN}S^{MN} = 0 \tag{8.15}$$

Comparison of (8.13) with (8.15) shows that the application of the simplified yield condition (8.11) changes the loading criterion in such a way that the term $2S^{KL}\dot{E}_{LN}$ is neglected in the sum $G_{LN}\dot{S}^{KL} + 2S^{KL}\dot{E}_{LN}$. This can be done only if $2S^{KL}\dot{E}_{LN} \ll G_{LN}\dot{S}^{KL}$. However, for perfectly plastic material it can be shown [23], [24] that both terms are of the same order of magnitude. Therefore, such a simplification affecting the loading criterion may change the behaviour of the material after yielding. The predicted behaviour of shell structure after reaching the yield point load may be changed completely as a result of such an improper simplification if material of the shell is assumed to be perfectly plastic. Therefore the existing solutions for perfectly plastic shells at large deflections and small strains [25], [26] must not be accepted without caution and, sometimes, suitable verification.

The detailed analysis of this problem exceeds the scope of the paper.

Let us now consider the transformation of the yield condition (8.11) into the space of generalized stresses defined by the equations (6.8), (6.9). To this end, we first transform the yield condition (8.11) to be expressed in terms of the modified stress tensor $\hat{S}^{\Delta\Gamma}$ defined by eq. (6.4). Then we obtain the yield condition in the form

$$F = 3G_{\Delta\phi}G_{\Gamma\theta}\hat{S}^{\Delta\Gamma}\hat{S}^{\theta\phi} - (G_{\Delta\Gamma}\hat{S}^{\Delta\Gamma})^2 - 2\sigma_o^2 = 0 \tag{8.16}$$

which coincides with (8.11) provided $S^{33} = 0$.

To perform further transformation a relation between stresses and strain rates is needed. Assuming the plastic potential flow law as associated with the yield condition (8.16), the strain rate $\dot{E}_{KL}$ is defined by the relation

$$\dot{E}_{\Delta\Gamma} = \nu \frac{\partial F}{\partial \hat{S}_{\Delta\Gamma}} = 2\nu(3G_{\Delta\phi}G_{\Gamma\theta}\hat{S}^{\phi\theta} - G_{\phi\theta}G_{\Delta\Gamma}\hat{S}^{\phi\theta}) \tag{8.17}$$

where the flow multiplier $\nu \geq 0$.

The relation (8.17) can be inversed to obtain the stresses,

$$\hat{S}^{\phi\theta} = \frac{1}{6\nu}(\dot{E}_{\Delta\Gamma}G^{\phi\Delta}G^{\theta\Gamma} + \dot{E}^{\Delta}_{\Delta}G^{\phi\theta}) \ . \tag{8.18}$$

Substitution of the stress from (8.18) into the yield condition (8.16) allows to evaluate the flow mulitplier

$$\nu = \frac{1}{2\sqrt{6}\,\sigma_o}(\dot{E}^{\Delta}_{\Gamma}\dot{E}^{\Gamma}_{\Delta} + \dot{E}^{\Delta}_{\Delta}\dot{E}^{\Gamma}_{\Gamma})^{1/2} \ . \tag{8.19}$$

Let us notice that, in view of (8.19), the stress components (8.18) are fully determined by the deformation mode and are homogeneous of degree zero with respect to the strain rates, thus also with respect to time. This rate (or time) independence is one of the specific features of the plastic deformation process.

In order to obtain the yield condition (8.16) in terms of stress resultants it is necessary to integrate the stress components (8.18) over the shell thickness.

To this end, substituting (4.10) into (8.18), we express the stress tensor $\hat{S}^{\phi\theta}$ by means of the deformation rates of the middle surface

$$\hat{S}^{\phi\theta} = \frac{1}{6\nu}\,[(\dot{\Lambda}^{\phi\theta} + x^3\dot{K}^{\phi\theta}) + (\dot{\Lambda}^{\Delta}_{\Delta} + x^3\dot{K}^{\Delta}_{\Delta})G^{\phi\theta}] \;. \tag{8.20}$$

Putting (8.20) into the definition (6.8), (6.9) of the stress resultants in a shell, the following expressions are obtained:

$$N^{\Delta\Gamma} = \int_{-H}^{+H} \frac{1}{6\nu}[(\dot{\Lambda}^{\Delta\Gamma} + \dot{\Lambda}^{\phi}_{\phi}G^{\Delta\Gamma}) + x^3(\dot{K}^{\Delta\Gamma} + \dot{K}^{\phi}_{\phi}G^{\Delta\Gamma})]dx^3 \;, \tag{8.21}$$

$$M^{\Delta\Gamma} = \int_{-H}^{+H} \frac{1}{6\nu}\,[x^3(\dot{\Lambda}^{\Delta\Gamma} + \dot{\Lambda}^{\phi}_{\phi}G^{\Delta\Gamma}) + (x^3)^2(\dot{K}^{\Delta\Gamma} + \dot{K}^{\phi}_{\phi}G^{\Delta\Gamma})]dx^3 \;. \tag{8.22}$$

Under the assumption (3.1) $\mu^{\Delta}_{\Gamma} = \delta^{\Delta}_{\Gamma}$ and $G^{\Delta\Gamma} = A^{\Delta\Gamma}$, then the integrals in (8.21) and (8.22) can be evaluated and the stress resultants $N^{\Delta\Gamma}$, $M^{\Delta\Gamma}$ rewritten in the form:

$$N^{\Delta\Gamma} = (\dot{\Lambda}^{\Delta\Gamma} + \dot{\Lambda}^{\phi}_{\phi}A^{\Delta\Gamma})I_1 + (\dot{K}^{\Delta\Gamma} + \dot{K}^{\phi}_{\phi}A^{\Delta\Gamma})I_2 \;, \tag{8.23}$$

$$M^{\Delta\Gamma} = (\dot{\Lambda}^{\Delta\Gamma} + \dot{\Lambda}^{\phi}_{\phi}A^{\Delta\Gamma})I_2 + (\dot{K}^{\Delta\Gamma} + \dot{K}^{\phi}_{\phi}A^{\Delta\Gamma})I_3 \tag{8.24}$$

where

$$I_i = \frac{1}{6}\int_{-H}^{+H} \frac{(x^3)^{i-1}}{\nu}\,dx^3 \;, \quad i = 1,2,3 \;. \tag{8.25}$$

The integrals I_i can be easily evaluated since in view of (8.19) and (4.10) - (4.12) they reduce to the algebraic integrals of the type $\int (x^3)^{i-1}[A(x^3)^2 + Bx^3 + C]^{-1/2}\,dx^3$.

Since all the terms in (8.23), (8.24) are homogeneous of order zero with respect to the strain rates, the set of six functions $N^{\Delta\Gamma}$, $M^{\Delta\Gamma}$ depends on five independent parameters only and thus represents a parametric form of the Huber-Mises yield condition for uniform shells.

The strain rates $\dot{\Lambda}_{\Delta\Gamma}$, $\dot{K}_{\Delta\Gamma}$ can be determined from the relations (8.23), (8.24). However, they are specified uniquely only if the determinant

$$\Delta = I_1 I_3 - (I_2)^2 \neq 0 \quad . \tag{8.26}$$

Whenever $\Delta = 0$ the singular points or lines exist on the yield surface.

In general, the shell geometry enters the yield equations (8.23), (8.24) through the metric $A^{\Delta\Gamma}$. In the simplest approximations, however, we consider a shell to be locally flat. Then the yield surface does not depend on the shell geometry.

The above explained procedure of derivation of the yield condition for shells is essentially due to Ilyushin [17], [27].

8.2.2. Huber-Mises condition for cylindrical shell

The yield condition in the general form (8.23), (8.24) is highly nonlinear and involved. However, for specific geometry of a shell the yield condition may be considerably simplified as the result of projection or intersection of the general form.

Let us illustrate this fact by an example of a cylindrical shell subjected to rotationally symmetric deformation process. The principal stress and strain rate directions coincide then with the lines of principal curvatures and therefore the following components vanish:

$$N^{12} = M^{12} = \dot{\Lambda}_{12} = \dot{K}_{12} = 0 \quad . \tag{8.27}$$

For short cylindrical shells the circumferential rate of curvature may also be considered as negligibly small

$$\dot{K}_{22} \simeq 0 \quad . \tag{8.28}$$

If the cylindrical shell is simply supported and transmits no end laods in the axial direction then in view of equilibrium eq. (7.10) we have

$$N^{11} = 0 \ . \tag{8.29}$$

Vanishing of the axial force and the circumferential rate of curvature applied into (8.23) results in the condition

$$[\dot{\Lambda}^{11} + (\dot{\Lambda}^1_1 + \dot{\Lambda}^2_2)A^{11}]I_1 + (\dot{K}^{11} + \dot{K}^1_1 A^{11})I_2 = 0 \ . \tag{8.30}$$

In the cylindrical frame of reference the nonvanishing components of the first and the second fundamental forms of cylindrical surface of radius R are:

$$A^{11} = A_{11} = 1 \quad , \quad A_{22} = R^2 \quad , \quad A^{22} = \frac{1}{R^2} \quad , \tag{8.31}$$

$$B_{22} = -R \quad , \quad B^2_2 = -\frac{1}{R} \ . \tag{8.32}$$

In view of (8.31) and (8.25), the equation (8.30) leads to the requirements

$$(2\dot{\Lambda}^{11} + R^2\dot{\Lambda}^{22})I_1 = 0 \quad , \tag{8.33}$$

$$2\dot{K}^{11}I_2 = 0 \tag{8.34}$$

and therefore

$$\dot{\Lambda}^{11} = -\frac{R^2}{2}\dot{\Lambda}^{22} \quad , \quad I_2 = 0 \ . \tag{8.35}$$

Using (8.27), (8.28), (8.31) and (8.35) in (8.23) and (8.24), the generalized stresses are found to be

$$N^{22} = \frac{3}{2}\dot{\Lambda}^{22}I_1 \quad , \tag{8.36}$$

$$M^{11} = 2\dot{K}^{11}I_3 \quad , \tag{8.37}$$

$$M^{22} = \frac{1}{2R^2}M^{11} \ . \tag{8.38}$$

Making use of (8.28) and $(8.35)_1$ in (8.19), the flow parameter may be expressed in the form

$$\nu = \frac{1}{2\sqrt{3}\,\sigma_o} \left[(\dot{E}_1^1)^2 + (\dot{E}_2^2)^2 + \dot{E}_1^1 \dot{E}_2^2\right]^{1/2} =$$

$$= \frac{1}{2\sqrt{3}\,\sigma_o} \left[(3(\dot{\Lambda}_2^2)^2 + 4(x^3)^2(\dot{K}_1^1)^2\right]^{1/2} \quad . \tag{8.39}$$

Using (8.39) in (8.25), the integrals I_1 and I_3 can be evaluated and next substituted into (8.36) - (8.37). This leads to the following dimensionless stress resultants.

$$n_2 = \frac{1}{2}\,\eta \ln \frac{\sqrt{1+\eta^2}+1}{\sqrt{1+\eta^2}-1} \quad , \tag{8.40}$$

$$m_1 = \frac{2}{\sqrt{3}} \left(\sqrt{1+\eta^2} - \frac{\eta^2}{2} \ln \frac{\sqrt{1+\eta^2}+1}{\sqrt{1+\eta^2}-1}\right) , \tag{8.41}$$

$$m_2 = \frac{1}{2}\, m_1 \tag{8.42}$$

where

$$n_2 = \frac{N_2^2}{2\sigma_o H} \quad , \quad m_1 = \frac{M_1^1}{\sigma_o H^2} \quad , \quad m_2 = \frac{M_2^2}{\sigma_o H^2} \quad , \tag{8.43}$$

$$\eta = \frac{\sqrt{3}}{2H} \frac{\dot{\Lambda}_2^2}{\dot{K}_1^1} \tag{8.44}$$

The equations (8.40) - (8.42) constitute the parametric form of the Huber-Mises yield condition for a cylindrical uniform shell. The obtained condition is an intersection of the general yield locus (8.23), (8.24) by the hyperplanes $N_{12} = 0$, $M_{12} = 0$, $N_{11} = 0$ projected orthogonally onto the plane (M_{11}, N_{22}) due to the kinematical requirements $\dot{K}_{12} = 0$, $\dot{\Lambda}_{12} = 0$, $\dot{K}_{22} = 0$. The interaction curve on the plane $m_1 - n_2$ is plotted by solid line in the Fig. 8.1. It is close to an ellipse with semi axes $(1, \frac{2}{\sqrt{3}})$ marked by broken line.

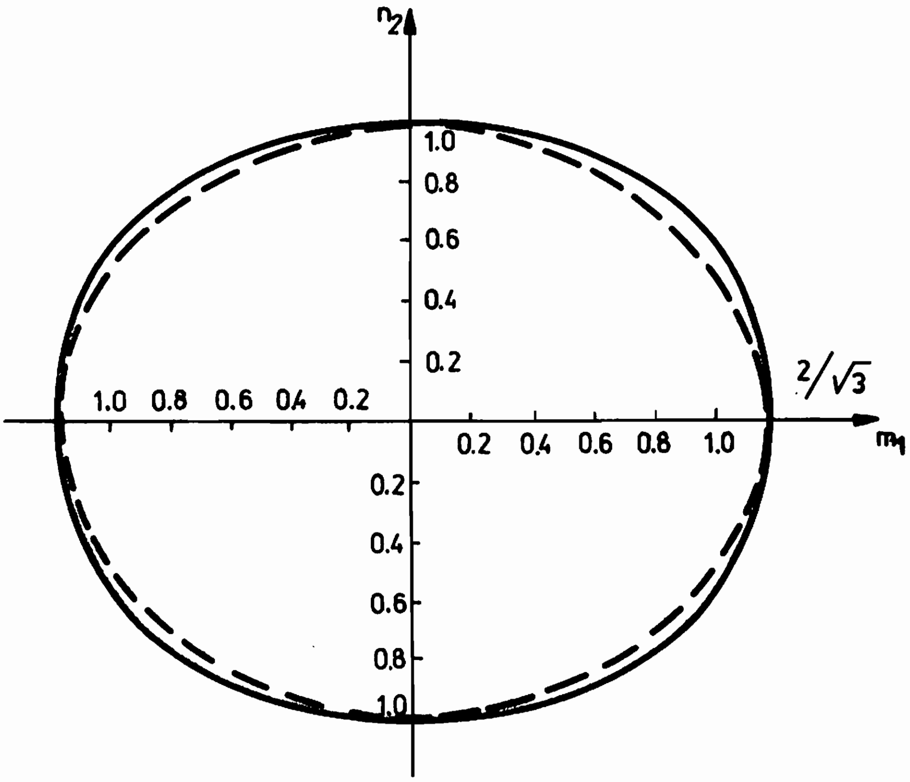

Fig. 8.1

A detailed derivation of the Huber-Mises yield condition for rotationally symmetric uniform shells is given in [19], [20], [29], [30].

8.3. Tresca yield condition for uniform shell

8.3.1. General form

The Tresca yield condition for a uniform shell was first obtained by Onat and Prager [31]. The procedure can effectively be applied to any yield condition which is piece-wise linear in terms of the principal stresses.

The Tresca yield condition states that the maximum shearing is equal to or less than half of the tensile yield stress. In the multiaxial stress state with the principal stresses σ_1, σ_2, σ_3 the Tresca yield condition can be written as follows

$$\max.\ (|\sigma_1 - \sigma_3| \ ,\ |\sigma_2 - \sigma_3| \ ,\ |\sigma_1 - \sigma_2|) \le \sigma_o \ . \qquad (8.45)$$

The transformation of the yield condition (8.45) into the space of the modified second Piola-Kirchhoff stress tensor $\hat{S}^{KL}$ can be done similarly as in the section 8.2 for Huber-Mises yield condition. Analogous simplifications lead, as before, to the yield condition in the same form

$$\max.\ (|\hat{S}_1| \ ,\ |\hat{S}_2| \ ,\ |\hat{S}_1 - \hat{S}_2|) \le \sigma_o \ . \qquad (8.46)$$

The condition (8.46) , visualized in Fig. 8.2, can be rewritten in the explicit form, convenient in further considerations.

$$F_1 = \hat{S}_1 - \sigma_o = 0 \ ,\quad F_2 = \hat{S}_2 - \sigma_o = 0 \ ,\quad F_3 = \hat{S}_1 - \hat{S}_2 - \sigma_o = 0$$

$$F_4 = \hat{S}_1 + \sigma_o = 0 \ ,\quad F_5 = \hat{S}_2 + \sigma_o = 0 \ ,\quad F_6 = \hat{S}_1 - \hat{S}_2 + \sigma_o = 0. \qquad (8.47)$$

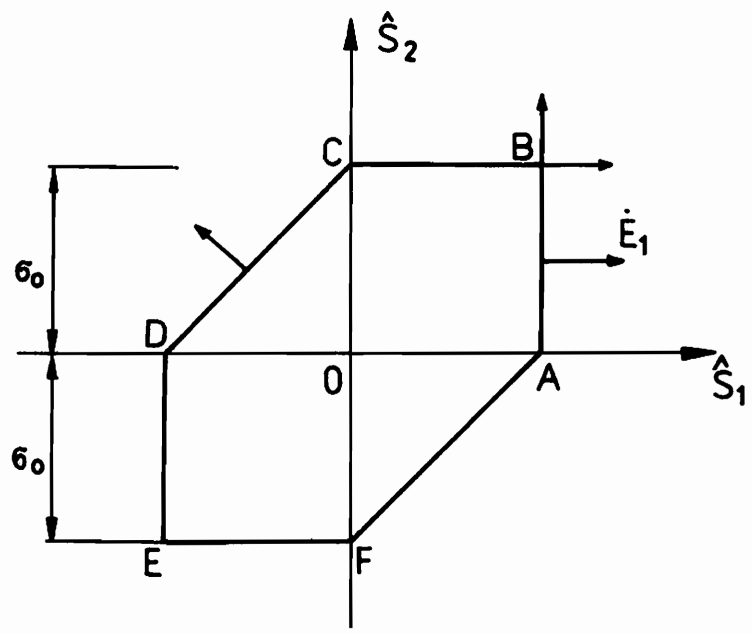

Fig. 8.2

In order to specify the stress distribution over the shell thickness and hence to find the interaction of stress resultant at yielding of a cross-section, let us perform the mapping of stress regimes represented by the sides of the hexagon ABCDEF (Fig. 8.2) onto the principal strain rate plane. To this end we make use of the plastic potential flow law which can be expressed as

$$\dot{E}_\Delta = \nu_K \frac{\partial F_K}{\partial \hat{S}_\Delta} \quad , \quad \nu_K \geq 0 \quad , \quad \Delta = 1,2 \quad , \quad K = 1,2\ldots6 \ . \tag{8.48}$$

It follows from (8.48) that for the side AB, analytically described by the equation

$$F_1 = \hat{S}_1 - \sigma_o = 0 \quad , \tag{8.49}$$

we have

$$\dot{E}_1 = \nu_1 \geq 0 \quad , \quad \dot{E}_2 = 0 \quad . \tag{8.50}$$

The equations (8.50) are represented, on the strain rate plane, by the coordinate line $\dot{E}_1$ shown in Fig. 8.3.

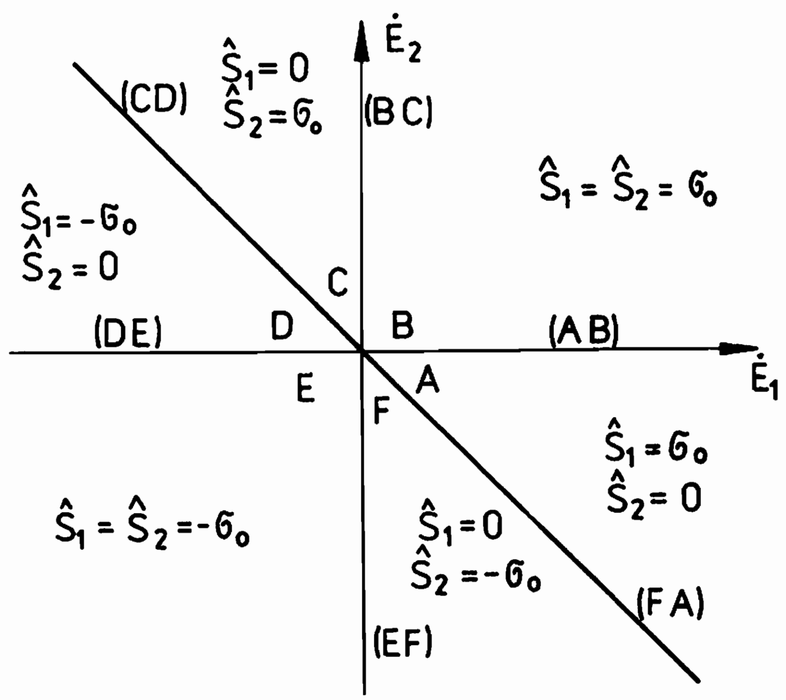

Fig. 8.3

For the stress state $\hat{S}_1 = \hat{S}_2 = \sigma_o$ represented by the corner B, the equations $F_1 = 0$ and $F_2 = 0$ must be satisfied simultaneously. Hence, in view of (8.47) and (8.48), we have

$$\dot{E}_1 = \nu_1 \geq 0 \quad , \quad \dot{E}_2 = \nu_2 \geq 0 \quad . \tag{8.51}$$

The corner B is mapped, therefore, into the quadrant of positive $\dot{E}_1$, $\dot{E}_2$ on the $\{\dot{E}_1, \dot{E}_2\}$ plane (Fig. 8.3). Similarly, the remaining sides and corners of the hexagon are mapped as straight lines and regions shown in the Fig. 8.3.

In view of (4.10) and (3.5) the strain rate distribution over the thickness of a shell is

$$\begin{aligned} \dot{E}_1 &= \dot{\Lambda}_1 + \xi H \dot{K}_1 \quad , \\ \dot{E}_2 &= \dot{\Lambda}_2 + \xi H \dot{K}_2 \quad , \\ \dot{E}_3 &= -(\dot{\Lambda}_1 + \dot{\Lambda}_2) - \xi H(\dot{K}_1 + \dot{K}_2) \end{aligned} \tag{8.52}$$

where $\xi = \frac{X^3}{H}$ is a dimensionless normal coordinate. For points belonging to a shell we clearly have $-1 \leq \xi \leq 1$.

Denoting ξ_1, ξ_2, ξ_3 the dimensionless normal coordinates for which strain rates $\dot{E}_1$, $\dot{E}_2$, $\dot{E}_3$ are respectively equal to zero, we obtain from (8.52) the following relations

$$\xi_1 = -\frac{\dot{\Lambda}_1}{H\dot{K}_1} \quad , \quad \xi_2 = -\frac{\dot{\Lambda}_2}{H\dot{K}_2} \quad , \quad \xi_3 = \frac{-\dot{\Lambda}_1 - \dot{\Lambda}_2}{H(\dot{K}_1 + \dot{K}_2)} \quad . \tag{8.53}$$

A typical distribution of strain rates is shown in Fig. 8.4. For definiteness, let us consider the case when

$$-1 \leq \xi_1 < \xi_3 < \xi_2 \leq 1 \tag{8.54}$$

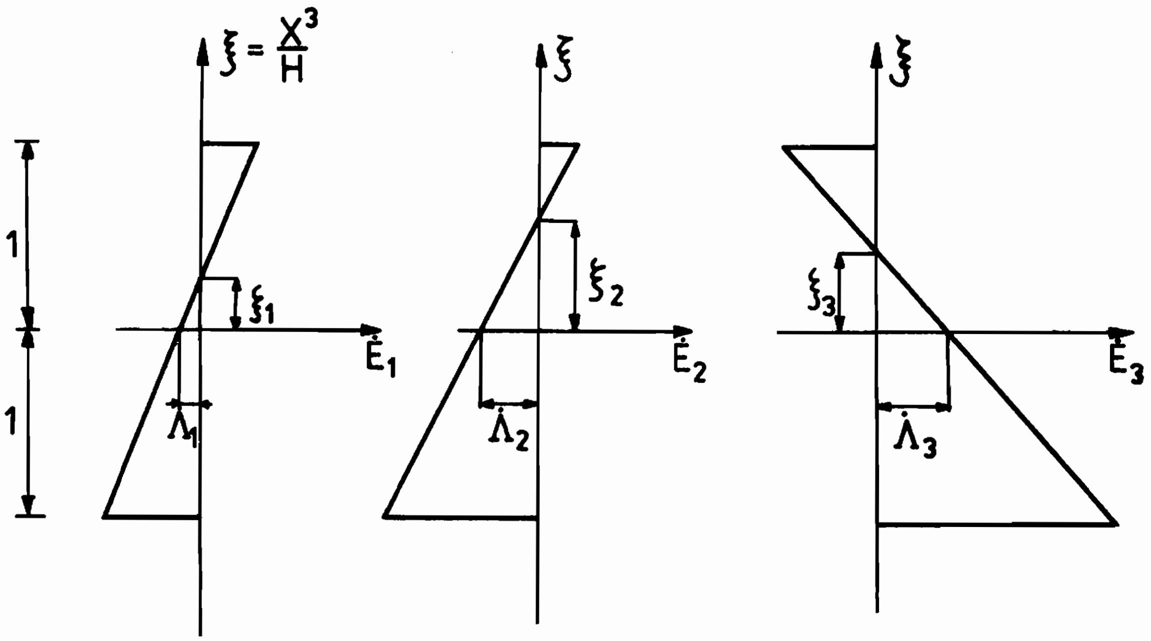

Fig. 8.4

Then, for $\xi_2 \le \xi \le 1$ the strain rates $\dot{E}_1$ and $\dot{E}_2$ are positive and from the mapping given in the Fig. 8.3 we have $\hat{S}_1 = \hat{S}_2 = \sigma_o$. Analogous considerations applied to the remaining values of ξ lead to the results listed in the Table 8.1

Table 8.1

$\xi_2 \le \xi \le 1$	$\dot{E}_1 > 0$, $\dot{E}_2 \ge 0$	$\hat{S}_1 = \hat{S}_2 = \sigma_o$	point B in Fig. 8.2
$\xi_3 \le \xi \le \xi_2$	$\dot{E}_1 > 0$, $\dot{E}_2 \le 0$, $\frac{\dot{E}_2}{\dot{E}_1} \ge -1$	$\hat{S}_1 = \sigma_o$, $\hat{S}_2 = 0$	point A
$\xi_1 \le \xi \le \xi_3$	$\dot{E}_1 \ge 0$, $\dot{E}_2 < 0$, $\frac{\dot{E}_2}{\dot{E}_1} \le -1$	$\hat{S}_1 = 0$, $\hat{S}_2 = -\sigma_o$	point F
$-1 \le \xi \le \xi_1$	$\dot{E}_1 \le 0$, $\dot{E}_2 < 0$	$\hat{S}_1 = \hat{S}_2 = -\sigma_o$	point E

The stress distribution in the considered case is shown in Fig. 8.5.

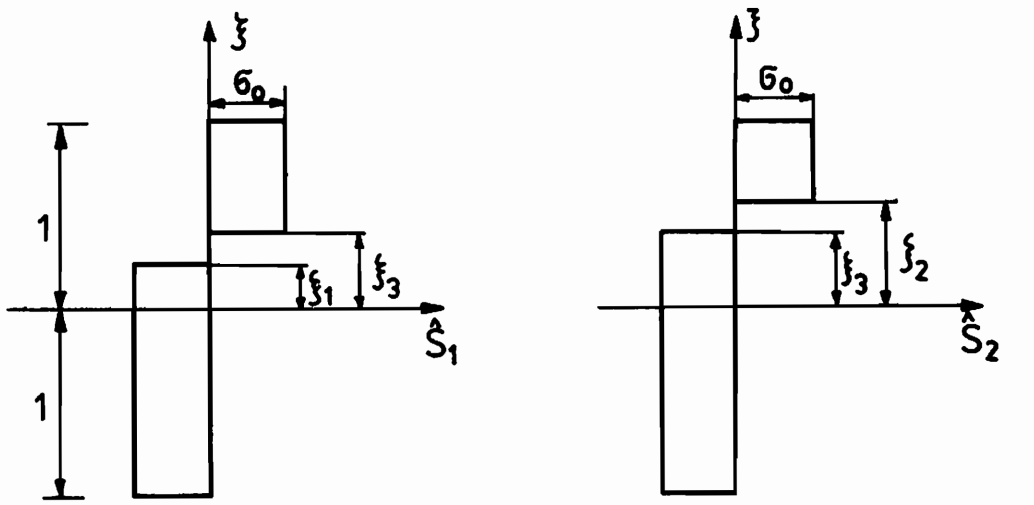

Fig. 8.5

In view of (6.8), (6.9) and (6.11) the dimensionless stress resultants and bending moments are:

$$n_1 = -\frac{1}{2}(\xi_1 + \xi_3) \quad , \qquad m_1 = 1 - \frac{1}{2}(\xi_1^2 + \xi_3^2) \quad ,$$
$$n_2 = -\frac{1}{2}(\xi_2 + \xi_3) \quad , \qquad m_2 = 1 - \frac{1}{2}(\xi_2^2 + \xi_3^2) \quad . \tag{8.55}$$

In particular situations any of the parameters ξ_1, ξ_2, ξ_3 can be an intermediate one and then each of the others can be the largest (or the smallest). There are clearly six possible arrangements of these points corresponding to six yield hypersurfaces. Equations of these hypersurfaces are assembled in Table 8.2.

Table 8.2

Intermediate parameter	Hyper-surface	Stress resultants			
		n_1	n_2	m_1	m_2
ξ_1	$F_1^{\pm}$	$\mp\frac{1}{2}(\xi_1+\xi_3)$	$\pm\frac{1}{2}(\xi_2-\xi_3)$	$\pm 1\mp\frac{1}{2}(\xi_1^2+\xi_3^2)$	$\pm\frac{1}{2}(\xi_2^2-\xi_3^2)$
ξ_3	$F_{12}^{\pm}$	$\mp\frac{1}{2}(\xi_1+\xi_3)$	$\mp\frac{1}{2}(\xi_2+\xi_3)$	$\pm 1\mp\frac{1}{2}(\xi_1^2+\xi_3^2)$	$\pm 1\mp\frac{1}{2}(\xi_3^2+\xi_2^2)$
ξ_2	$F_2^{\pm}$	$\mp\frac{1}{2}(\xi_3-\xi_1)$	$\mp\frac{1}{2}(\xi_2+\xi_3)$	$\pm\frac{1}{2}(\xi_1^2-\xi_3^2)$	$\pm 1\mp\frac{1}{2}(\xi_3^2+\xi_2^2)$

Since all the parameters ξ_1, ξ_2, ξ_3 should be within the interval (-1,1), hence if one or more is greater than 1, then that parameter must be replaced by 1 in the Tabel 8.2; similarly, if a parameter is less than -1 it must be replaced by -1. Such replacements do not change the stress distribution in the cross-section. For example, for $\xi_2 > 1$, $0 \le \xi_3 \le 1$ in Fig. 8.4 the stress distribution in Fig. 8.5 is the same as for $\xi_2 = 1$ $0 \le \xi_3 \le 1$. Kinematically it means that the flow mechanism is not uniquely specified. Such situations occur at singular points of the yield hypersurface.

If any of the parameters ξ_1, ξ_2, ξ_3 becomes indeterminate as computed from eqs (8.53), then the others must be equal to each other. Consider, for example, the stress regimes AB and DE of the yield hexagon shown in the Fig. 8.2.

Then

$$\dot{E}_2 = \dot{\Lambda}_2 + \xi H\dot{K}_2 = 0 \quad \text{for any } \xi \text{ from the interval } (-1, +1) \tag{8.56}$$

and therefore

$$\dot{\Lambda}_2 = 0 \quad , \quad \dot{K}_2 = 0 \; . \tag{8.57}$$

Substitution of (8.57) into (8.53) yields $\xi_1 = \xi_3$ and ξ_2 is indeterminate. The appropriate yield hypersurface is obtained when ξ_1 is set equal to ξ_3 in either of the first two lines of Table 8.2.

$$n_1 = \pm\xi_1 \quad , \quad m_1 = \pm 1 \mp (\xi_1)^2 \quad . \tag{8.58}$$

Eliminating ξ_1 from eqs (8.58), we obtain

$$H_1^{\pm} : \quad m_1 = \pm(1 - n_1^2) \tag{8.59}$$

which is represented by hypercylinders parallel to the directions n_2 and m_2 in the space n_1, n_2, m_1, m_2. Similarly, if $\xi_2 = \xi_3$, either of the last two lines of the Table 8.2 leads to

$$H_2^{\pm} : \quad m_2 = \pm(1 - n_2^2) \tag{8.60}$$

Finally, if $\xi_1 = \xi_2$, the first and the third lines show that

$$H_{12}^{\pm} : \quad m_1 - m_2 = \pm[1 - (n_1 - n_2)^2] . \qquad (8.61)$$

All six hypercylindrical parts of the yield locus are written in Table 8.3.

Table 8.3

Coincidence	Deformation mode	Hypersurface	Yield condition
$\xi_1 = \xi_3$	$\dot{\Lambda}_2 = 0$, $\dot{K}_2 = 0$	$H_1^{\pm}$	$m_1 = \pm (1 - n_1^2)$
$\xi_1 = \xi_2$	$\dot{\Lambda}_1 = -\dot{\Lambda}_2$, $\dot{K}_1 = -\dot{K}_2$	$H_{12}^{\pm}$	$m_1 - m_2 = \pm[1-(n_1 - n_2)^2]$
$\xi_2 = \xi_3$	$\dot{\Lambda}_1 = 0$, $\dot{K}_1 = 0$	$H_2^{\pm}$	$m_2 = \pm(1 - n_2^2)$

The yield hypersurfaces $F_1^{\pm}$, $F_{12}^{\pm}$, $F_2^{\pm}$, $H_1^{\pm}$, $H_{12}^{\pm}$, $H_2^{\pm}$ bound a convex region in the four-dimensional space (n_1, n_2, m_1, m_2) which contains all the statically admissible states of stress resultants.

If one of the generalized stresses (n_1, n_2, m_1, m_2) or generalized strain rates $(\dot{\Lambda}_1, \dot{\Lambda}_2, \dot{K}_1, \dot{K}_2)$ is equal to zero, the yield condition can be visualized either by its intersection with a hyperplane or by its projection onto the specified subspace. The yield condition is then an ordinary surface in the three-dimensional space.

8.3.2. Particular cases

Two important particular situations occur when either one of the axial forces or one of the bending moments can be eliminated.

1°. Consider first the case when $n_2 = 0$, then from the Table 8.3 we have

Table 8.4

$\xi_1 = \xi_3$	$H_1^{\pm}$	$m_1 = \pm (1 - n_1^2)$
$\xi_1 = \xi_2$	$H_{12}^{\pm}$	$m_1 - m_2 = \pm (1 - n_1^2)$
$\xi_2 = \xi_3$	$H_2^{\pm}$	$m_2 = \pm 1$

Putting $n_2 = 0$ in the Table 8.2 we have $\xi_2 = \xi_3$ and, eliminating ξ_3 and ξ_1 from the remaining relations, we obtain

$$F^+_{12} : \quad m_1 = 1 - 2[(n_1 + \frac{\sqrt{1-m_2}}{2})^2 + \frac{1-m_2}{4}] \; ,$$

$$F^+_2 : \quad m_1 = 2[(n_1 + \frac{\sqrt{1-m_2}}{2})^2 - \frac{1-m_2}{4}] \; . \tag{8.62}$$

The yield surface for $n_2 = 0$ is visualized in the Fig. 8.6.

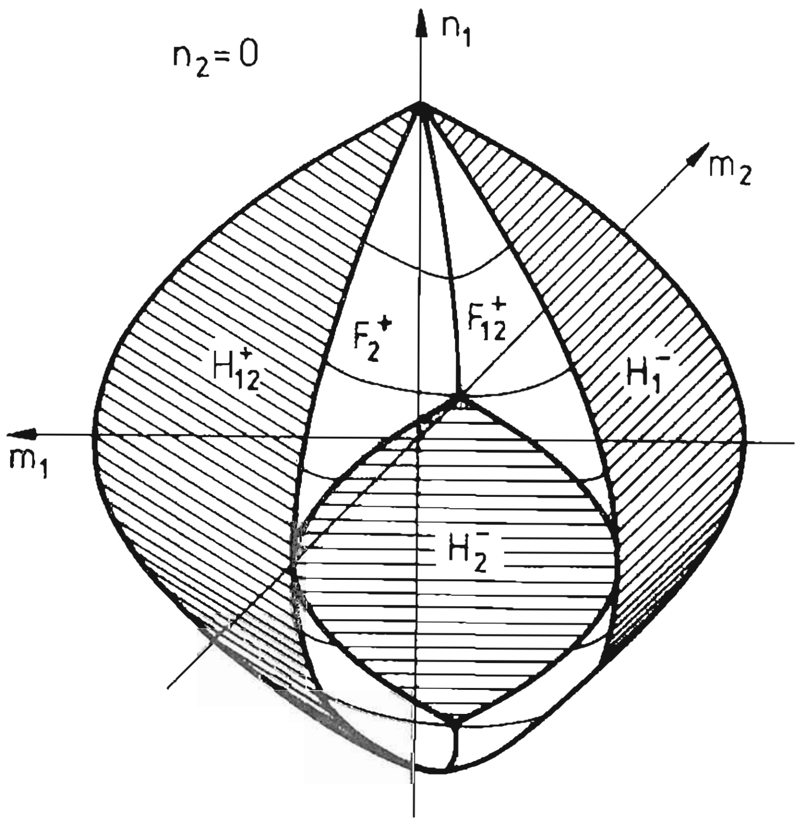

Fig. 8.6

2^o. Another interesting case occurs when $\dot{K}_2 = 0$. Then m_2 may be treated as a reaction. Such a situation takes place when a short cylindrical shell is considered under axisymmetrical loading ($\dot{K}_2$ denotes here the circumferential rate of curvature).

In view of $(8.53)_2$ we have $\xi_2 \to \pm\infty$. Setting $\xi_2 = \pm 1$ in Table 8.2 and eliminating m_2, we obtain the yield surface described by the equations:

$$
\begin{aligned}
I^{\pm} &: \quad n_2 = \pm 1 \qquad \text{(planes)} \, , \\
II^{\pm} &: \quad n_2 - n_1 = \pm 1 \ \text{(planes)} \, , \\
III^{\pm} &: \quad m_1 = \pm(1 - n_1^2)\,\text{(parablolic cylinders)} \, , \\
IV^{\pm} &: \quad m_1 = \pm \frac{1}{2}\,[2 - (2n_2 - 1)^2 - (2n_2 - 2n_1 - 1)^2]\ \text{(paraboloids)} .
\end{aligned} \tag{8.63}
$$

The yield surface for $\dot{K}_2 = 0$ is visualized in Fig. 8.7.

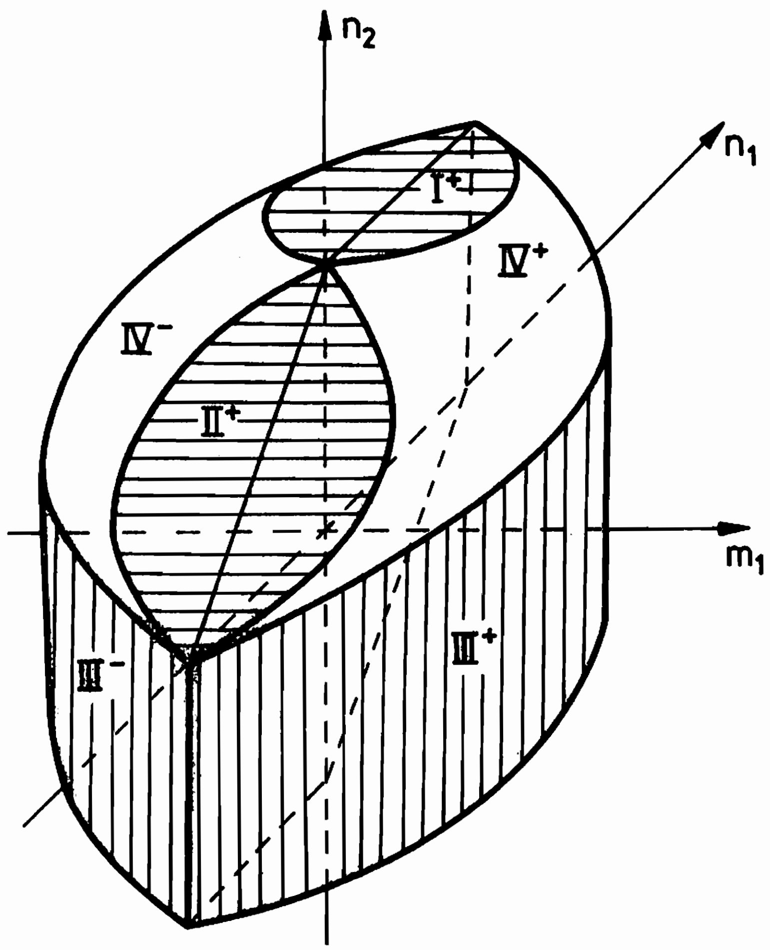

Fig. 8.7

8.4. Sandwich shells

To simplify the yield condition a uniform cross-section shell can be approximated by an idealized sandwich cross-section shell with the same resistance to pure tension and pure bending.

An ideal sandwich shell is composed of two thin sheets, each of thickness T and the tensile yield stress σ_o', separated by a core of thickness 2H' and no tensile strength. The sheets are so thin that the stress variation across each sheet can be neglected. Under these assumptions the yield constraints of a sandwich cross-section shell are

$$N_o = 2\sigma_o' T \quad , \quad M_o = 2\sigma_o' H' T \ . \tag{8.64}$$

The corresponding values for a uniform shell of thickness 2H are

$$N_o = 2\sigma_o H \quad , \quad M_o = \sigma_o H^2 \ . \tag{8.65}$$

Thus the sandwich shell is equivalent to the uniform shell if

$$\sigma_o' T = \sigma_o H \quad , \quad H' = \frac{1}{2} H \ . \tag{8.66}$$

Denoting by $\overset{+}{S}{}^{\Delta\Gamma}$ and $\overset{-}{S}{}^{\Delta\Gamma}$ the stresses in the outer and inner sheets of the sandwich shell, the stress resultants become:

$$N^{\Delta\Gamma} = (\overset{+}{S}{}^{\Delta\Gamma} + \overset{-}{S}{}^{\Delta\Gamma})T \quad , \quad M^{\Delta\Gamma} = (\overset{+}{S}{}^{\Delta\Gamma} - \overset{-}{S}{}^{\Delta\Gamma})TH' \ . \tag{8.67}$$

Solving Eqs (8.67) for the stresses and introducing the dimensionless stress resultants from Eq. (6.11) (where N_o, M_o are given by (8.64), we obtain

$$\begin{aligned} \overset{+}{S}{}^{\Delta\Gamma} &= \sigma_o'(n^{\Delta\Gamma} - m^{\Delta\Gamma}) \ , \\ \overset{-}{S}{}^{\Delta\Gamma} &= \sigma_o'(n^{\Delta\Gamma} + m^{\Delta\Gamma}) \ . \end{aligned} \tag{8.68}$$

8.4.1. Huber-Mises yield condition for rotationally symmetric sandwich shells

The Huber-Mises yield condition (8.11) written in terms of the principal stresses $\hat{S}_1$ and $\hat{S}_2$ takes form

$$\hat{S}_1^2 + \hat{S}_1\hat{S}_2 + \hat{S}_2^2 = \sigma_o'^2 \tag{8.69}$$

For a sandwich cross-section two yield conditions have to be considered, each corresponding to yielding of a single layer. Two cases are possible: either both sheets are plastic or only one yields. In the first case two plasticity conditions, obtained by substitution of (8.68) into (8.69), are satisfied simultaneously:

$$\begin{aligned} F^+ &= (n_1 + m_1)^2 - (n_1 + m_1)(n_2 + m_2) + (n_2 + m_2)^2 - 1 = 0 \\ F^- &= (n_1 - m_1)^2 - (n_1 - m_1)(n_2 - m_2) + (n_2 - m_2)^2 - 1 = 0 \end{aligned} \tag{8.70}$$

The stress point is then on the intersection of the two hypersurfaces (8.70), and the strain rate vector may be any linear combination with positive coefficient of the two vectors normal to each of the hypersurfaces (Fig. 8.8)

$$\begin{aligned} \dot{\lambda}_\Delta &= \nu^+ \frac{\partial F^+}{\partial n_\Delta} + \nu^- \frac{\partial F^-}{\partial n_\Delta} \quad , \\ \dot{\kappa}_\Delta \equiv H\dot{K}_\Delta &= \nu^+ \frac{\partial F^+}{\partial m_\Delta} + \nu^- \frac{\partial F^-}{\partial m_\Delta} \quad . \end{aligned} \tag{8.71}$$

In the other case, the equality sign must hold in one of the equations (8.70) and the other becomes an inequality.

For a rotationally symmetric deformation of cylindrical shells the change of circumferential curvature can be neglected, $\dot{K}_2 = 0$, and the circumferential moment m_2 can be eliminated from (8.70). Geometrically, this operation may be regarded as a projection of the four-dimensional region onto the hyperplane m_2 = const. Then yield condition (8.70) can

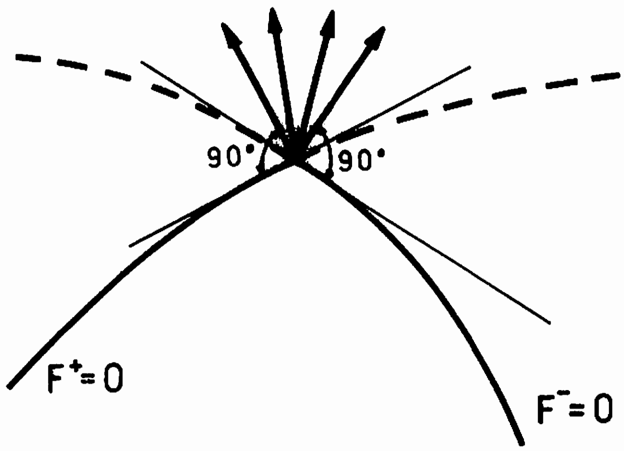

Fig. 8.8

be written in the three-dimensional space in the form

$$(n_1^2 - n_1 n_2 + n_2^2)\ [1 + \frac{3m_1^2}{(2n_2 - n_1)^2}] - 1 = 0 \quad ,$$

$$m_2 = \frac{m_1(n_2 - 2n_1)}{2n_2 - n_1} \quad . \tag{8.72}$$

For a particular case of no axial load, we have $n_1 = 0$ and the Eqs (8.72) reduce to the form

$$n_2^2 + \frac{3}{4} m_1^2 - 1 = 0 \ ,$$

$$m_2 = \frac{1}{2} m_1 \quad . \tag{8.73}$$

The equation $(8.73)_1$ describes the ellipse wnich is shown in Fig. 8.1 by broken line.

8.4.2. Tresca yield condition for rotationally symmetric sandwich shells

In terms of the principal stresses $\hat{S}_1$ and $\hat{S}_2$ the Tresca yield condition takes the form (8.47). The six equalities must be satisfied by the stresses in the outer and inner sheets. Therefore, the Tresca yield condition for the sandwich shell consists of twelve expressions obtained by introducing in (8.47) the stresses expressed in terms of the stress resultants as given in (8.68).

The resulting expressions are listed in Table 8.5 as derived by Hodge [19].

Table 8.5

Hyper-plane	Equation of hyper-plane	Range of validity
F_1^+	$n_1 - m_1 = 1$	$0 \le n_2 - m_2 \le 1$
F_2^+	$n_2 - m_2 = 1$	$0 \le n_1 - m_1 \le 1$
F_3^+	$-n_1 + m_1 + n_2 - m_2 = 1$	$0 \le n_2 - m_2 \le 1$ or $0 \le -n_1 + m_1 \le 1$
F_4^+	$-n_1 + m_1 = 1$	$0 \le -n_2 + m_2 \le 1$
F_5^+	$-n_2 + m_2 = 1$	$0 \le -n_1 + m_1 \le 1$
F_6^+	$n_1 - m_1 - n_2 + m_2 = 1$	$0 \le n_1 - m_1 \le 1$ or $0 \le -n_2 + m_2 \le 1$
F_1^-	$n_1 + m_1 = 1$	$0 \le n_2 + m_2 \le 1$
F_2^-	$n_2 + m_2 = 1$	$0 \le n_1 + m_1 \le 1$
F_3^-	$-n_1 - m_1 + n_2 + m_2 = 1$	$0 \le n_2 + m_2 \le 1$ or $0 \le -n_1 - m_1 \le 1$
F_4^-	$-n_1 - m_1 = 1$	$0 \le -n_2 - m_2 \le 1$
F_5^-	$-n_2 - m_2 = 1$	$0 \le -n_1 - m_1 \le 1$
F_6^-	$n_1 + m_1 - n_2 - m_2 = 1$	$0 \le n_1 + m_1 \le 1$ or $0 \le -n_2 - m_2 \le 1$

9. THEORY OF LIMIT ANALYSIS

9.1. Statement of the problem

Let us consider a shell subjected to a system of loads that increase *quasistatically* and *in proportion*. The term in proportion indicates that the ratio of intensities of any two loads remains constant during the loading process, so the load is fully prescribed over the shell by a single *loading parameter* p which is called an *intensity of loading*.

As p slowly increases starting from zero, it will first reach a value p_e for which the yield condition is satisfied ($F = 0$) at some points of the shell; for $p > p_e$ a region will develop in which $F = 0$. However, the plastic domain is not, in general, free to deform since the enclosing rigid (or elastic) portion of the shell is strong enough to restrain it from plastic motion. As p is further increased, the plastic region will continue to grow until for $p = p_o$ the rigid portion becomes insufficient to restrain the plastic region from motion and the shell starts to deform plastically. Such a load is called a *yield-point load* or *limit load.*

It can be shown [33] that the yield-point load p_o for an idealized rigid plastic shell has the same value as for elastic-plastic shell, provided the effects due to geometry changes are ignored.

9.2. Theorems of limit analysis

A solution to a limit analysis problem is termed complete when it provides:

1^o. intensity of limit load p_o,
2^o. resulting stress distribution at the limit load,
3^o. mechanism of motion at the limit load

and is such that the following relations are statisfied:

1^o. yield condition,
2^o. equilibrium equations,
3^o. geometrical relations (strain rates-velocities),
4^o. flow law (normality condition),

5^{o}. stress and velocity boundary conditions,

6^{o}. non-negativity of the internal dissipation energy.

In many shell problems it is difficult to find a complete solution of the limit analysis. In such cases the two theorems of limit analysis provide very convenient tool to obtain solutions that satisfy only some of the requirements 1^{o} - 6^{o} and give the bounds on p_o from below and above.

The statement of the lower and the upper bound theorems will be more convenient after defining statically and kinematically admissible states.

A statically admissible stress field is, by definition, a field of generalized stresses $\underset{\sim}{Q}^{-}$: $\{n^{-}_{\Delta\Gamma}, m^{-}_{\Delta\Gamma}\}$ such that

1^{o}. the equilibrium equations and the stress boundary conditions are satisfied, and

2^{o}. the yield condition is not violated, i. e. $F(\underset{\sim}{Q}^{-}) \le 0$.

Each of the fields $\underset{\sim}{Q}^{-}$ corresponds to a certain intensity of loading which will be denoted by p^{-}.

A kinematically admissible strain rate field is a field of generalized strain rates $\underset{\sim}{q}^{*}$: $\{\dot{\lambda}^{*}_{\Delta\Gamma}, \dot{\kappa}^{*}_{\Delta\Gamma}\}$ such that

1^{o}. it can be derived from a velocity field $\underset{\sim}{v}^{*}$ which satisfies the velocity boundary conditions,

2^{o}. the external energy rate $\dot{D}^{*}_{ext.}$ caused by the applied loads $\underset{\sim}{p}$ on the assumed velocities $\underset{\sim}{v}^{*}$ is positive,

$$\dot{D}^{*}_{ext.} = \int_{S_p} \underset{\sim}{p} \cdot \underset{\sim}{v}^{*} \, d S_p > 0 \qquad (9.1)$$

where S_p is a part of the surface S where external load is applied.

The kinematically admissible stress field $\underset{\sim}{Q}^{*}$ is determined by the kinematically admissible strain rate vector $\underset{\sim}{q}^{*}$ by means of the flow law. Finally, a load $\underset{\sim}{p}^{*}$ is assessed from the requirement

$$\int_{S_p} \underset{\sim}{p}^* \cdot \underset{\sim}{v}^* dS_p = \int_{S} \underset{\sim}{Q}^* \cdot \underset{\sim}{q}^* dS \tag{9.2}$$

Lower-bound theorem says that the limit load p_o *is the largest of all loads* p^- *corresponding to statically admissible stress fields*

$$p^- \leq p_o \tag{9.3}$$

To prove the lower-bound theorem we consider the actual stress tensor $\underset{\sim}{Q}$ as a vector from the origin to the point on the yield surface $F = 0$ in the stress space. Then the actual strain rate vector q is directed along a normal to the yield surface at that point (Fig. 9.1)

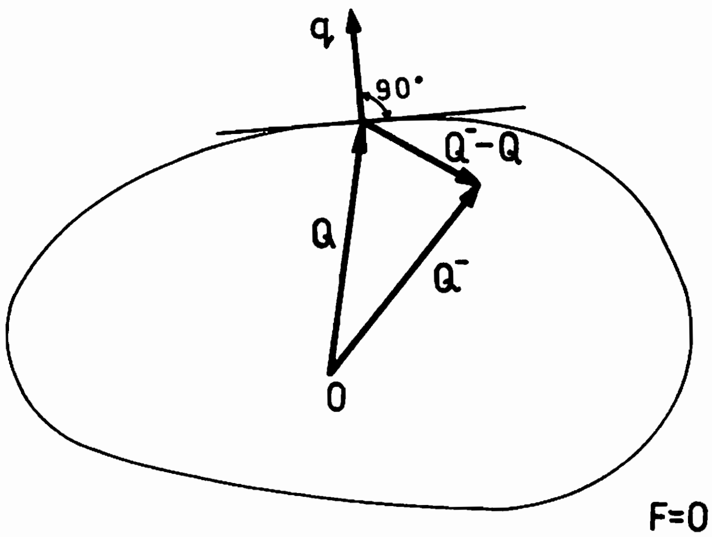

Fig. 9.1

Since $\underset{\sim}{Q}^-$ is a statically admissible stress vector, its point lies on or within the yield surface, and the vector $(\underset{\sim}{Q}^- - \underset{\sim}{Q})$ makes an angle equal to or greater than 90^o with the strain rate vector $\underset{\sim}{q}$. Therefore

$$(\underset{\sim}{Q}^- - \underset{\sim}{Q}) \cdot \underset{\sim}{q} \leq 0 \ . \tag{9.4}$$

Finally, since (9.4) must be satisfied at each point of the shell, we may write

$$\int_S \underset{\sim}{Q}^- \cdot \underset{\sim}{q}\, dS \leq \int_S \underset{\sim}{Q} \cdot \underset{\sim}{q}\, dS \quad . \tag{9.5}$$

By principle of virtual power we have

$$\int_S \underset{\sim}{Q}^- \cdot q \, dS = \int_{S_p} \underset{\sim}{p}^- \cdot \underset{\sim}{v} \, dS_p \tag{9.6}$$

and

$$\int_S \underset{\sim}{Q} \cdot \underset{\sim}{q} \, dS_p = \int_{S_p} \underset{\sim}{p}_o \cdot \underset{\sim}{v} dS_p \tag{9.7}$$

where $\underset{\sim}{v}$ is the actual velocity field.

Substitution of (9.6) and (9.7) into (9.5) leads to the inequality

$$\int_{S_p} \underset{\sim}{p}^- \cdot v dS_p \leq \int_{S_p} \underset{\sim}{p}_o \cdot \underset{\sim}{v} dS_p \quad . \tag{9.8}$$

Since the systems of loads differ only by a positive scalar factor, the inequality (9.8) furnishes (9.3).

Upper-bound theorem says, that the limit load p_o *is the smallest of all loads* p^* *corresponding to a kinematically admissible mechanisms,*

$$p^* \geq p_o \quad . \tag{9.9}$$

Similarly as before we can show (Fig. 9.2) that the following inequality takes place

$$(\underset{\sim}{Q} - \underset{\sim}{Q}^*) \cdot q^* \leq 0 \quad . \tag{9.10}$$

Next, since (9.10) must be satisfied at each point of the shell we may write

$$\int_S \underset{\sim}{Q} \cdot \underset{\sim}{q}^* dS \leq \int_S \underset{\sim}{Q}^* \cdot q^* dS \quad . \tag{9.11}$$

By the principle of virtual power we have

$$\int_S \underset{\sim}{Q} \cdot \underset{\sim}{q}^* dS = \int_{S_p} \underset{\sim}{p}_o \cdot v^* dS_p \quad . \tag{9.12}$$

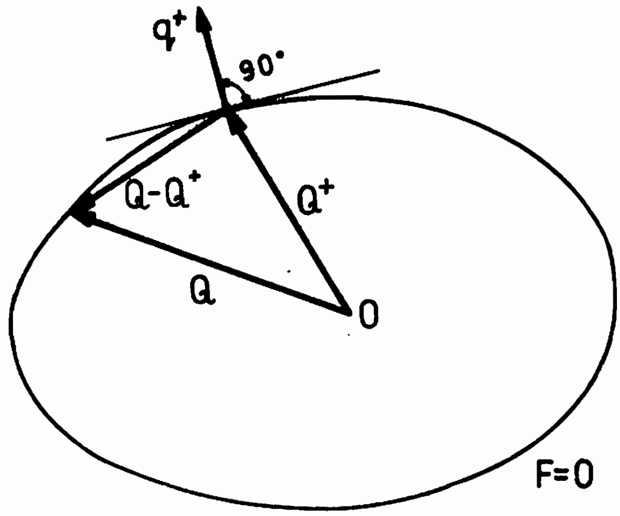

Fig. 9.2

Substitution of (9.2) and (9.12) into (9.11) leads to the inequality

$$\int_{S_p} \underset{\sim}{p}_o \cdot \underset{\sim}{v}^* dS_p \le \int_{S_p} \underset{\sim}{p}^* \cdot \dot{v}^* dS_p \quad . \tag{9.13}$$

Again, since the systems of loads differ only by positive scalar factor, the inequality (9.13) furnishes (9.9).

The inequalities (9.3) and (9.9) may be combined to yield the upper and the lower bounds on the actual yield point load,

$$p^- \le p_o \le p^* \quad . \tag{9.14}$$

In terms of the introduced definitions we can state that *a solution is complete if and only if it is both statically and kinematically admissible.*

Since the fundamental theorems of limit analysis are based exclusively on the concept of statically admissible stress fields and kinematically admissible strain rate fields, they are valid for both idealizations of the material; rigid-plastic and elastic-plastic. In the case of elastic-plastic material, however, the assumption on small elastic and elastic-plastic strains has to be remembered.

The lower- and the upper-bound theorems of limit analysis were first given by Gwozdew [33], and later independently proved by Drucker, Greenberg and Prager [35], [34] and Hill [36].

9.3. Bounding surface lemma

As we have seen in the chapter 8, the actual yield conditions for shells are nonlinear and, therefore, lead to considerable mathematical difficulties. Various approximate yield loci are known in the literature [19], [20], [32]. One method of simplifying the problem is to linearize the yield condition.

Suppose that the smooth curve in Fig. 9.3 represents an exact yield condition $F = 0$.

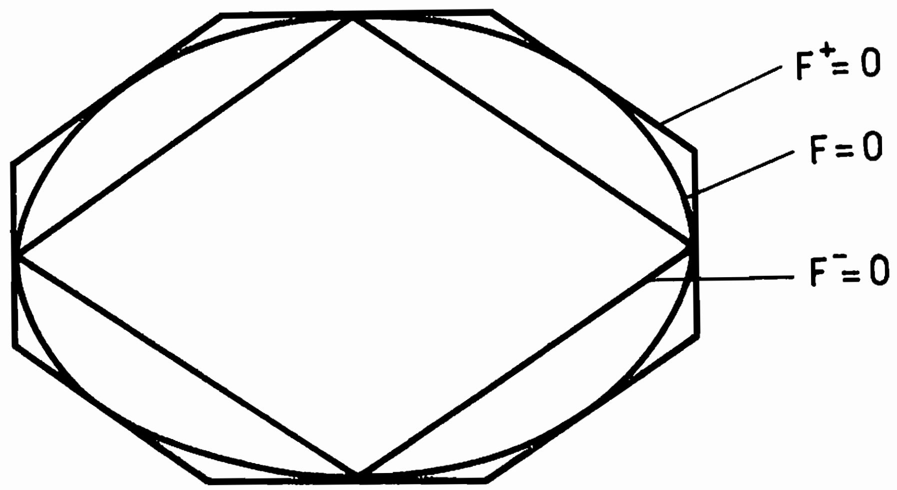

Fig. 9.3

It can be approximated in many ways. However, using an inscribed $F^- = 0$ and a circumscribed $F^+ = 0$ yield criterion, we are able, by means of the limit analysis theorems, to bound the error introduced by the approximation.

Let us denote p_o^-, p_o, p_o^+ the yield-point loads of the complete solutions corresponding, respectively, to the yield conditions $F^- = 0$, $F = 0$, $F^+ = 0$. Then the stress distribution corresponding to the complete solution according to the yield condition $F^- = 0$ will be only statically admissible according to the exact curve $F = 0$. Similarly, the stress distribution corresponding to the exact curve $F = 0$ will be only statically admissible according to the yield condition $F^+ = 0$. Hence, by the lower-bound theorem

$$p_o^- \le p_o \quad , \quad p_o \le p_o^+ \quad . \tag{9.15}$$

These inequalities may be combined to yield the upper and the lower bounds on the actual yield-point load p_o,

$$p_o^- \le p_o \le p_o^+ \quad . \tag{9.16}$$

If the outer and inner approximating surfaces are geometrically similar, as shown in Fig. 9.4, then the corresponding yield-point loads are similarly related as the surfaces themselves

$$p_o^- \le p_o \le \alpha p_o^- \quad \text{where} \quad \alpha \ge 1 \tag{9.17}$$

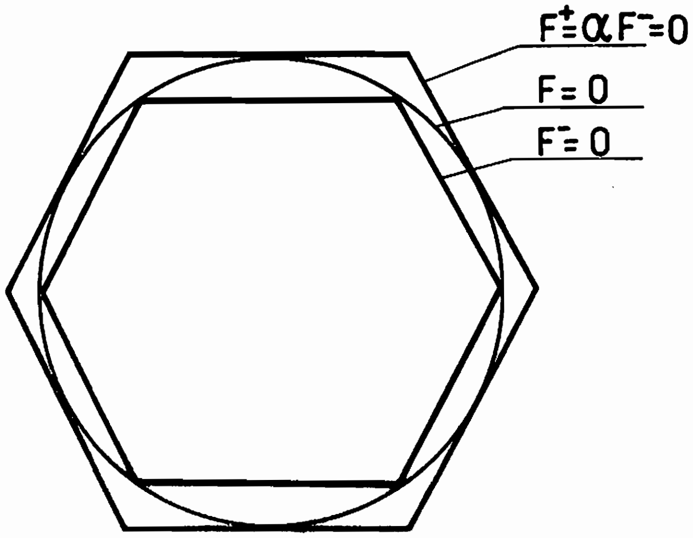

Fig.9.4

Sometimes it may be impossible to find a complete solution even of the approximate problem. In these cases the lower bound theorem can be applied to the interior yield surface and the upper bound theorem to the exterior surface. Then

$$(p_o^-)^- \le p_o^- \le p_o \le p_o^+ \le (p_o^+)^* \ . \tag{9.18}$$

When the two approximating curves are similar, (9.18) becomes

$$(p_o^-)^- \leq p_o^- \leq p_o \leq \alpha p_o^- \leq (\alpha p_o^-)^*. \tag{9.19}$$

Finally, it is worth noting that for a chosen yield criterion (e.g. Tresca or Huber-Mises) the yield surface for a sandwich cross-section shell lies inside the yield surface for an equivalent uniform shell.

10. PLASTIC ANALYSIS OF SHELLS AT LARGE DEFLECTIONS. SIMPLE EXAMLES

10.1. Statement of the problem

The deformation process of rigid-plastic or elastic-plastic shells may be roughly divided into three stages.

The first one is that of strains and displacements remaining small. The limit load theory furnishes a solution of the problem giving information about value of the yield-point load and the stress and strain rate fields if the complete solution is found.

At the second stage of the deformation process the moderate and large deflections develop, though strains usually remain small. The simplified, geometrically non-linear theories, presented in chapter 3, furnish the suitable strain-displacement relations.

At the third stage of the plastic deformation process large and unrestricted strains and displacements are encountered. Then, in view of a highly non-linear and involved form of fundamental relations, there are only numerical solutions that can be found. However, for some particular situations the simplifying assumptions can be made. For example, if during the deformation process the membrane forces only are essential, the influence of bending moments being neglected, the well developed membrane theory provides the required solution.

The approximate solutions known from the literature [25], [26], [39] - [41] indicate that in most of the practical cases the shallow shells reach the membrane state at the deflections of the order of magnitude of the shell thickness, whereas the quasi-shallow shells do so at the deflection ten times greater.

The analysis of the first stage of plastic deformation process and the third one in the case of membrane state are well developed in the literature [19], [30] - [38], [42] - [44]. The obtained solutions lead to the linear relationship between load and displacements. Therefore, presenting the analysis of deformation process of cylindrical and spherical shells attention will mainly be paid to the second stage of plastic deformation i. e. the deformation at moderately large and large deflections.

10.2. Cylindrical shell, rigid-plastic solution

10.2.1. Basic equations

Let us consider the behaviour of a rigid-perfectly plastic cylindrical shell closed at either end by a rigid plate and subjected to uniformly distributed internal dead load $\underset{\sim}{P}$ (Fig. 10.1).

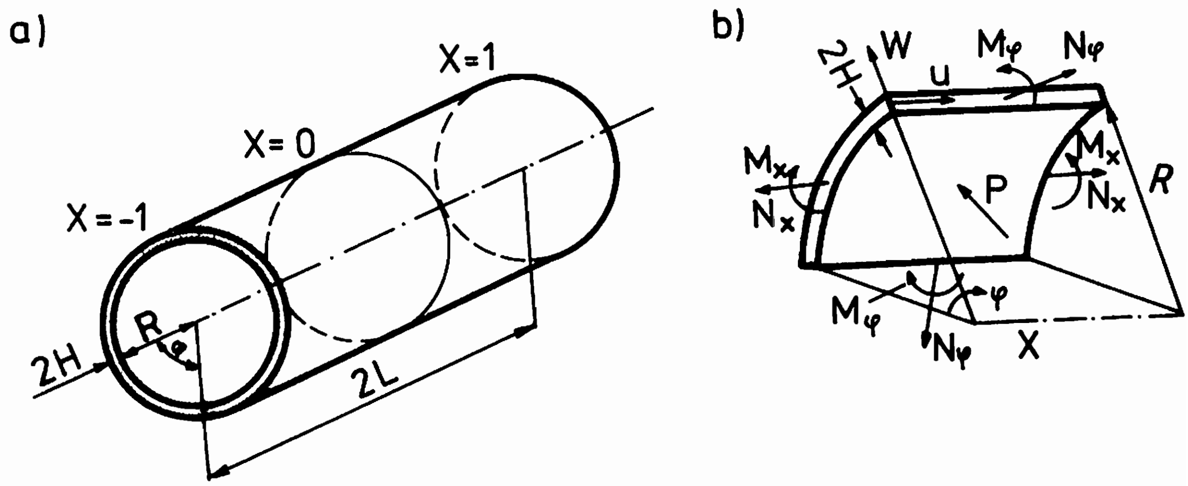

Fig. 10.1

The origin of cylindrical coordinate system (X, φ, R) will be conveniently placed at the centre of the shell.

The stress state in the wall is described by the stress resultants N^{xx}, $N^{\varphi\varphi}$ and stress couples M^{xx}, $M^{\varphi\varphi}$ in the axial and circumferential directions, respectively.

The kinematic behaviour of the shell, under the actual rotationally symmetric conditions of loading and deformation, is described by two components of the displacement vector: W in the radial direction and U in the X-direction .

Let us introduce the following dimensionless quantities:

$$w = \frac{W}{R} \quad , \quad u = \frac{U}{R} \quad , \quad x = \frac{X}{L} \quad , \quad \alpha = \frac{L^2}{2RH} \quad , \quad h = \frac{2H}{R} \quad , \tag{10.1}$$

$$\lambda_x = \Lambda_x^x \quad , \quad \lambda_\varphi = \Lambda_\varphi^\varphi \quad , \quad \kappa_x = HK_x^x \quad , \quad \kappa_\varphi = HK_\varphi^\varphi \quad , \tag{10.2}$$

$$n_x = \frac{N_x^x}{N_o} \quad , \quad n_\varphi = \frac{N_\varphi^\varphi}{N_o} \quad , \quad m_x = \frac{M_x^x}{M_o} \quad , \tag{10.3}$$

$$p = \frac{PR}{N_o} \tag{10.4}$$

where R, 2L, 2H are the shell radius, length and thickness, respectively, as indicated in Fig. 10.1.

Let us first consider the case of a short cylindrical shell ($L/R \simeq \simeq O(\varepsilon)$) at moderately large deflection W and small axial displacement U. We can use the Donnell-Vlasov geometrical relations as shown in the Table 5.1, case 1b:

$$\begin{aligned} \Lambda_{\Delta\Gamma} &= V_{\Delta|\Gamma} - B_{\Delta\Gamma}W + \frac{1}{2} W_{|\Delta}W_{|\Gamma} \quad , \\ K_{\Delta\Gamma} &= W_{|\Delta\Gamma} \; . \end{aligned} \tag{10.5}$$

For the cylindrical coordinate system we have:

$$A_{xx} = A^{xx} = 1 \; , \; A_{\varphi\varphi} = R^2 \; , \; A^{\varphi\varphi} = \frac{1}{R^2} \; , \; B_{\varphi\varphi} = -R \; , \; B_\varphi^\varphi = -\frac{1}{R} \; . \tag{10.6}$$

Making use of (10.1), (10.2) and (10.6) in (10.5), the geometrical relations for rotationally symmetric deformation of a cylindrical shell can be written in the form:

$$\begin{aligned} \lambda_x &= u' + \frac{1}{2} \left(\frac{R}{L} w' \right)^2 \quad , \quad \lambda_\varphi = -w \quad , \\ \kappa_x &= - \frac{w''}{2\alpha} \qquad\qquad\quad , \quad \kappa_\varphi = 0 \end{aligned} \tag{10.7}$$

where prime denotes partial differentiation with respect to x.

The equilibrium equations for the Donnell-Vlasov strain-displacements equations have been derived in the chapter 7, by means of the principle of virtual energy, in the form:

$$N^{\Delta\Gamma}_{|\Gamma} + P^{\Delta} = 0$$

$$N^{\Delta\Gamma}B_{\Delta\Gamma} + (N^{\Delta\Gamma}w_{|\Delta})_{|\Gamma} + M^{\Delta\Gamma}_{|\Delta\Gamma} + P^3 = 0 \quad . \tag{10.8}$$

In view of (10.3), (10.4), (10.6) and rotational symmetry (10.8) can be rewritten:

$$n'_x = 0 \ ,$$

$$n_\varphi - (\frac{R}{L})^2 n_x w'' + \frac{1}{2\alpha} m''_x - p = 0 \ . \tag{10.9}$$

In the considered case of the closed cylinder subject to uniform pressure P, the axial force is equal to $\pi R^2 P$. This force is carried by the shell in a uniform manner, so in view of (10.4) the value of the axial force at the edge is

$$N^x_x = \frac{\pi R^2 P}{2\pi R} = p\,\frac{N_o}{2} \tag{10.10}$$

and, nondimensionally

$$n_x = \frac{N^x_x}{N_o} = \frac{p}{2} \quad . \tag{10.11}$$

Since the axial equilibrium $(10.9)_1$ requires n_x to be constant along the shell, $n_x = \frac{p}{2}$ applies over the entire length of the shell and, therefore, the equilibrium equations (10.9) may be written in the form:

$$\frac{1}{2\alpha} m''_x + n_\varphi - (\frac{R}{L})^2 \frac{p}{2} w'' - p = 0 \quad ,$$

$$n_x = \frac{p}{2} \quad . \tag{10.12}$$

We assume one of the simplest, the Tresca-sandwich yield condition shown in the Table 8.5. Since $\dot{\kappa}_\varphi = 0$, m_φ is reduced to the status of a reaction and may be eliminated from the twelve equations of the Table 8.5. Geometrically, this operation is regarded as projecting the four-dimensional region onto the hyperplane m_φ = const., thus obtaining a yield surface in the three-dimensional space consisting of twelve planes as illustrated in Fig. 10.2:

$$I^{\pm} : \quad n_{\varphi} = \pm 1$$

$$II^{\pm} : \quad n_{\varphi} - n_x = \pm 1 , \tag{10.13}$$

$$III^{\pm} : - n_x + m_x = \pm 1 ,$$

$$IV^{\pm} : - n_x - m_x = \pm 1 , \qquad (10.13)$$

$$V^{\pm} : 2n_{\varphi} - n_x - m_x = \pm 2 ,$$

$$VI : 2n_{\varphi} - n_x + m_x = 2 .$$

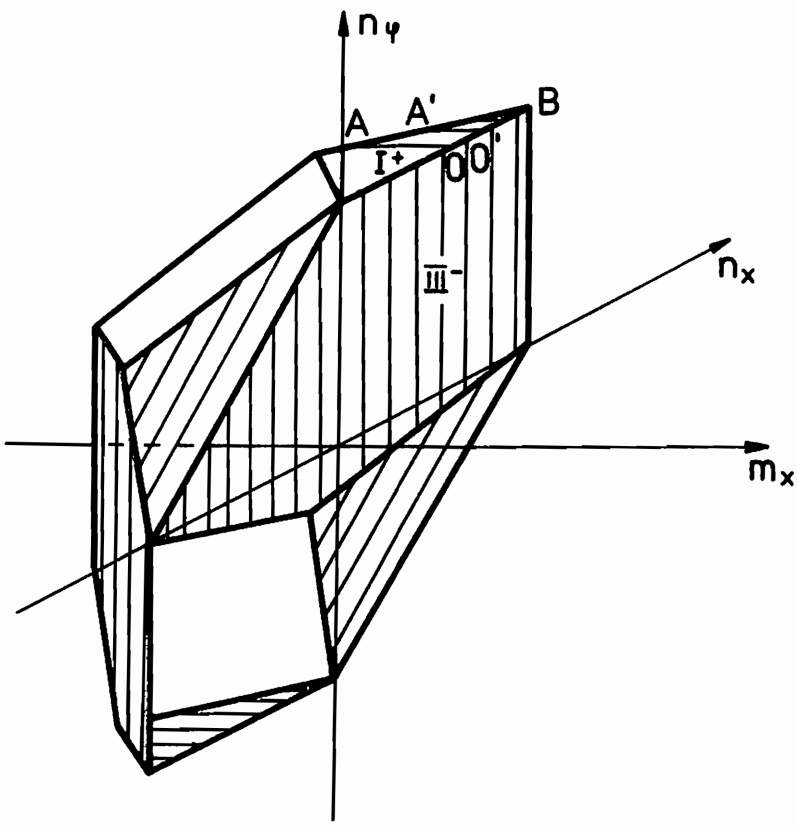

Fig. 10.2

The constancy of n_x along the shell means that the entire stress profile lies on an intersection of the yield surface with the plane $n_x = \frac{p}{2}$. Let us assume tentatively that $p \geq 1$; then

$$\frac{1}{2} \leq n_x \leq 1 \quad . \tag{10.14}$$

Finally, for internal pressure, n_φ is expected to be positive. Thus the entire stress profile must be on the face I^+,

$$I^+ : \quad n_\varphi = 1 \tag{10.15}$$

represented by the segment OA parallel to the m_x axis (Fig. 10.2).

10.2.2. Limit load solution

Since $w = 0$ at the yield-point load, eq. $(10.12)_1$ reduces to

$$\frac{1}{2\alpha} m''_x + n_\varphi - p = 0 \tag{10.16}$$

Substituting (10.15) into (10.16), the equilibrium equation takes the form

$$\frac{1}{2\alpha} m''_x + 1 - p = 0 \tag{10.17}$$

This equation must be solved under the boundary conditions:

$m_x(\pm 1) = 0$ at the simply supported ends, if diaphragms are attached by means of hinged joints *) , (10.18)

$m'_x(0) = 0$ at the centre. (10.19)

The solution is provided by

$$m_x = \alpha(p - 1)(x^2 - 1) \quad . \tag{10.20}$$

*) The case of clamped diaphragms is considered in [25], then $m_x(\pm 1) = 1 - n_x$.

The flow law associated with the yield condition (10.15) gives:

$$\dot{\lambda}_\varphi = \nu > 0 \quad , \quad \dot{\lambda}_x = 0 \quad , \quad \dot{\kappa}_x = 0 \quad , \quad \dot{\kappa}_\varphi = 0 \quad . \tag{10.21}$$

Differentiating $(10.7)_3$ with respect to time and then substituting the result into $(10.21)_3$, we obtain

$$\dot{\kappa}_x = - \frac{\dot{w}''}{2\alpha} = 0 \quad . \tag{10.22}$$

Therefore, the slope is constant in each half of the shell and a hinge circle develops at the centre $x = 0$. Hence, the centre must be on the intersections of faces I^+ and III^- (point A in Fig. 10.2)

$$\begin{aligned} I^+ &: \quad n_\varphi = 1 \quad , \\ III^- &: \quad n_x - m_x = 1 \quad . \end{aligned} \tag{10.23}$$

Substituting $n_x = \frac{p}{2}$ into (10.23), we get

$$m_x = \frac{p}{2} - 1 \qquad \text{for } x = 0 \quad . \tag{10.24}$$

The equations (10.24) and (10.20) for $x = 0$ determine the yield point load

$$p_o = 1 + \frac{1}{1 + 2\alpha} \tag{10.25}$$

10.2.3. Post yield behaviour

Whereas at the yield-point load the stress state is represented by the points on the segment AO (Fig. 10.2), the simplest hypothesis regarding the stress profile during the continuation of the yielding process is to assume that two zones can be distinguished in the shell:

1. Boundary zone $1 \geq x \geq \xi$ where

$$n_\varphi = 1 \quad , \quad n_x = \text{const.} \tag{10.26}$$

The stress profile is then represented by a segment A'O' parallel to the m_x axis.

2. Central zone $\xi \geq x \geq 0$ where

$$n_\varphi = 1 \quad , \quad n_x = \text{const.} \quad , \quad n_x - m_x = 1. \tag{10.27}$$

The stress profile is represented by a point A' on the segment AB (Fig. 10.2).

At the yield-point load the central zone is concentrated in the hinge circle in the middle of cylinder, hence $\xi = 0$. As the load increases, the central zone expands and $\xi \to 1$. The segment A'O' translates until point B is reached and the membrane state is attained.

In the boundary zone the fundamental relations are essentially the same as for the limit load analysis. However, since the boundary zone does not involve the centre of the shell, the boundary condition for $x = \pm 1$ can be applied solely to determine the constants of integration. Therefore, integration of eq. (10.17) with the boundary condition (10.18) furnishes

$$m_{x(1)} = \alpha(p - 1)(x^2 - 1) + B(t)(x - 1) \tag{10.28}$$

where B(t) is the constant of integration.

Similarly, integrating the eq. (10.22) with respect to time and x, and making use of the boundary and initial conditions:

$$\begin{aligned} w &= 0 \quad \text{for} \quad x = \pm 1 \quad , \\ w &= 0 \quad \text{for} \quad t = 0 \quad , \end{aligned} \tag{10.29}$$

we obtain the relations:

$$\begin{aligned} w''_{(1)} &= 0 \quad , \\ w_{(1)} &= A(t)(1 - x) \end{aligned} \tag{10.30}$$

where A(t) is the integration constant.

In the central zone the yield condition(the segment AB in Fig. 10.2) is given by the set of equations (10.23). The corresponding strain rate field obtained from the plastic flow law has the form

$$\dot{\lambda} = \nu_1 \geq 0 \quad , \quad \dot{\lambda}_x = \nu_2 \geq 0 \quad , \quad \dot{\kappa}_x = -\nu_2 \; . \tag{10.31}$$

These relations do not determine uniquely the direction of the strain rate vector. As the result of $(10.31)_2$ and $(10.31)_3$ we obtain

$$\dot{\lambda}_x = -\dot{\kappa}_x \quad . \tag{10.32}$$

Substitution of $(10.7)_1$ and $(10.7)_3$ differentiated with respect to time into (10.32) leads to the relation

$$\dot{u}'_{(2)} = -\left(\frac{R}{L}\right)^2 w'_{(2)}\dot{w}'_{(2)} + \frac{\dot{w}''}{2\alpha} \quad . \tag{10.33}$$

Equation (10.33) with the boundary conditions $u_{(2)}(0) = 0$ and $u_{(1)}(\xi) = u_{(2)}(\xi)$ constitute a Picard problem for a hyperbolic equation the solution of which is

$$u_{(2)} = u_{(1)}(\xi) + \int_x^{\xi}\int_0^{x}\left[-\frac{R}{\alpha} w'_{(2)}(\gamma,\eta)\dot{w}'_{(2)}(\gamma,\eta) + \frac{1}{\alpha} n_x(\eta)\dot{w}''_{(2)}(\gamma,\eta)\right]d\gamma d\eta \; . \tag{10.34}$$

From the equilibrium condition $(10.9)_1$ and the yield condition $(10.27)_3$ it follows that

$$m''_x = 0 \tag{10.35}$$

in the central zone.

Making use of (10.35) and the yield condition $(10.27)_1$ in the equilibrium equation $(10.12)_1$, we obtain

$$1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 pw'' - p = 0 \quad . \tag{10.36}$$

Integrating this equation twice with respect to x and making use of the boundary conditions:

$$w_{(2)}(0) = w_o \quad , \quad w'_{(2)}(0) = 0 \ , \tag{10.37}$$

we obtain the following relation for the deflection in the central zone of the shell:

$$w_{(2)} = \left(\frac{L}{R}\right)^2 \frac{(1-p)x^2}{p} + w_o \quad . \tag{10.38}$$

The relation (10.38) indicates that the generator of shell has, in the deformed state, the shape of a parabola. The bending moment $m_{x(2)}$ in the central zone is determined by the yield condition $(10.27)_3$.

Now, let us summarize the obtained results. The deflection and generalized stress fields are determined by the following set of equations:

$$\begin{aligned}
&w_{(1)} = A(t)(1-x) && \text{for} \quad 1 \geq x \geq \xi \ , \\
&w_{(2)} = \left(\frac{L}{R}\right)^2 \frac{(1-p)}{p} x^2 + w_o && \text{for} \quad \xi \geq x \geq 0 \ , \\
&m_{x(1)} = \alpha(p-1)(x^2-1) + B(t)(x-1) && \text{for} \quad 1 \geq x \geq \xi \ , \\
&m_{x(2)} = \frac{p}{2} - 1 && \text{for} \quad \xi \geq x \geq 0 \ , \\
&n_x = \frac{p}{2} \ , \\
&n_\varphi = 1 \quad .
\end{aligned} \tag{10.39}$$

In order to solve the problem, four quantities $A(t)$, $B(t)$, w_o, ξ must be determined in terms of p and x. To this end we use the continuity conditions for $x = \xi$:

$$w_{(1)} = w_{(2)} \quad , \quad w'_{(1)} = w'_{(2)} \quad , \quad m_{x(1)} = m_{x(2)} \quad , \quad m'_{x(1)} = m'_{x(2)} \quad .$$

The resulting equations have the form:

$$
\begin{aligned}
w_{(1)} &= 2\left(\frac{L}{R}\right)^2 \frac{(p-1)}{p}\,\xi(1-x) \;, \\
w_{(2)} &= \left(\frac{L}{R}\right)^2 \frac{(1-p)}{p}\,(x^2 - 2\xi + \xi^2) \;, \\
m_{x(1)} &= \alpha(p-1)[x^2 - 1 - 2\xi(x-1)] \;, \\
m_{x(2)} &= \frac{p}{2} - 1 \;, \\
n_x &= \frac{p}{2} \;, \\
n_\varphi &= 1 \;, \\
\xi &= 1 - \sqrt{\frac{2-p}{2\alpha(p-1)}} \quad \text{or} \quad p = 1 + \frac{1}{1 + 2\alpha(1-\xi)^2} \;, \\
w_o &= \left(\frac{L}{R}\right)^2 \frac{[p(1+2\alpha) - 2(1+\alpha)]}{2\alpha p} \;.
\end{aligned}
\tag{10.40}
$$

Setting $w_o = 0$ in eq. $(10.40)_8$ we clearly return to the relation (10.25) obtained as the solution of limit load problem. In view of relation $(10.40)_5$, increase of load p is accompanied by an increase of the axial force n_x. The membrane state is reached when

$$n_x = n_\varphi = 1 \quad , \quad m_x = 0 \quad . \tag{10.41}$$

The stress profile reduces then to the point B in Fig. 10.2. Making use of (20.41) in (10.40), we obtain

$$p = 2 \quad , \quad w = \frac{1}{2}\left(\frac{L}{R}\right)^2 (1 - x^2) \quad , \quad \xi = 1 \tag{10.42}$$

for which the membrane state is reached in the shell. The way in which the shell deforms at successive stages of plastic deformation until the membrane state is reached for $\xi = 1$ is shown in Fig. 10.3, illustrating the equations $(10.40)_1$, and $(10.40)_2$ for $\left(\frac{L}{R}\right)^2 = 0{,}1$ and $h = \frac{2H}{R} = 0{,}02$.

As it follows from Fig. 10.3 the membrane state is reached when the deflection w_o at the centre is 2,5 times greater than the shell thickness. In the case of longer shells, the attainment of membrane state

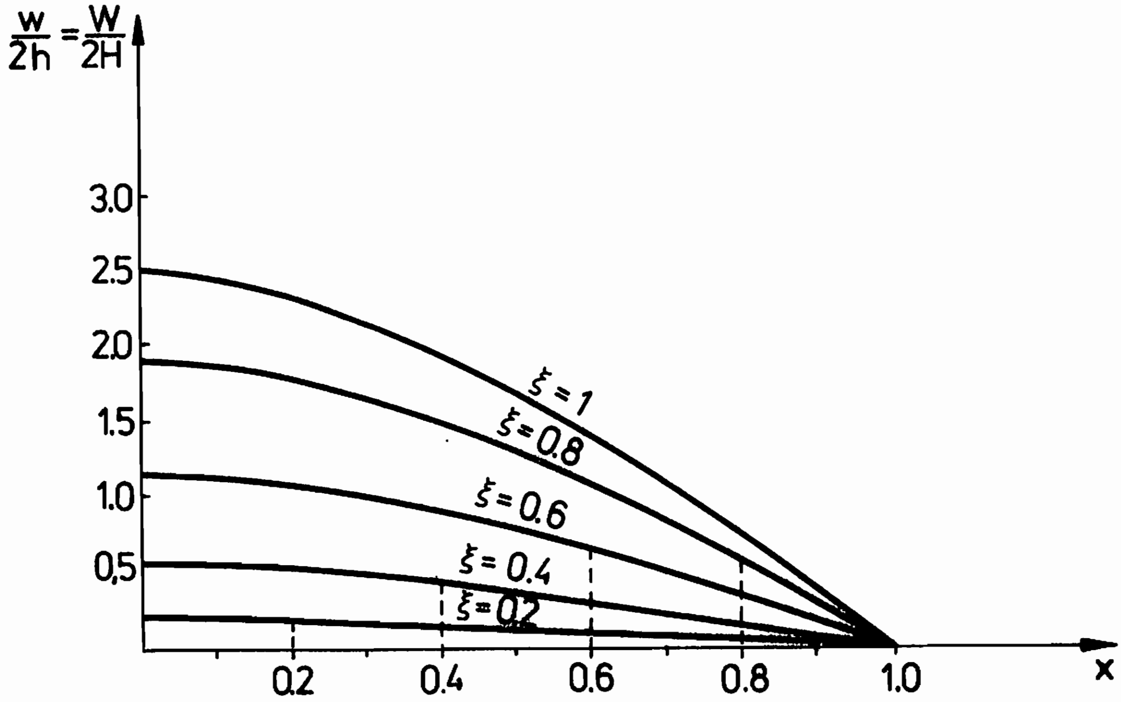

Fig. 10.3

is accompanied by a plastic deflection considerably greater. Then, the accuracy of the geometric relations and the equilibrium equations assumed in the present theory becomes insufficient and the theory of quasi-shallow shells at large defelctions has to be applied. The correct form of simplified geometrical relations is shown in the line 2c of Table 5.1 and proper equilibrium relations are derived in the chapter 7, eqs (7.17) - (7.18).

In such a formulation the considered problem was solved by M. Mitow and M. Duszek [41]. The procedure was analogous to the presented above, and therefore only the load - deflection relations are plotted in Fig. 10.4. by broken lines. For the sake of comparison, the resulting load-deflection relations given by Eq. $(10.40)_8$ are shown in the same figure by solid lines.

The difference between the deflections w_o obtained above according to the theory of shallow shells at moderately large deflections and obtained in [41] according to the theory of quasi-shallow shells at large deflections are roughly 5 % for $\frac{L^2}{R^2} = 0,1$ and roughly 15 % for $\frac{L^2}{R^2} = 0,4$.

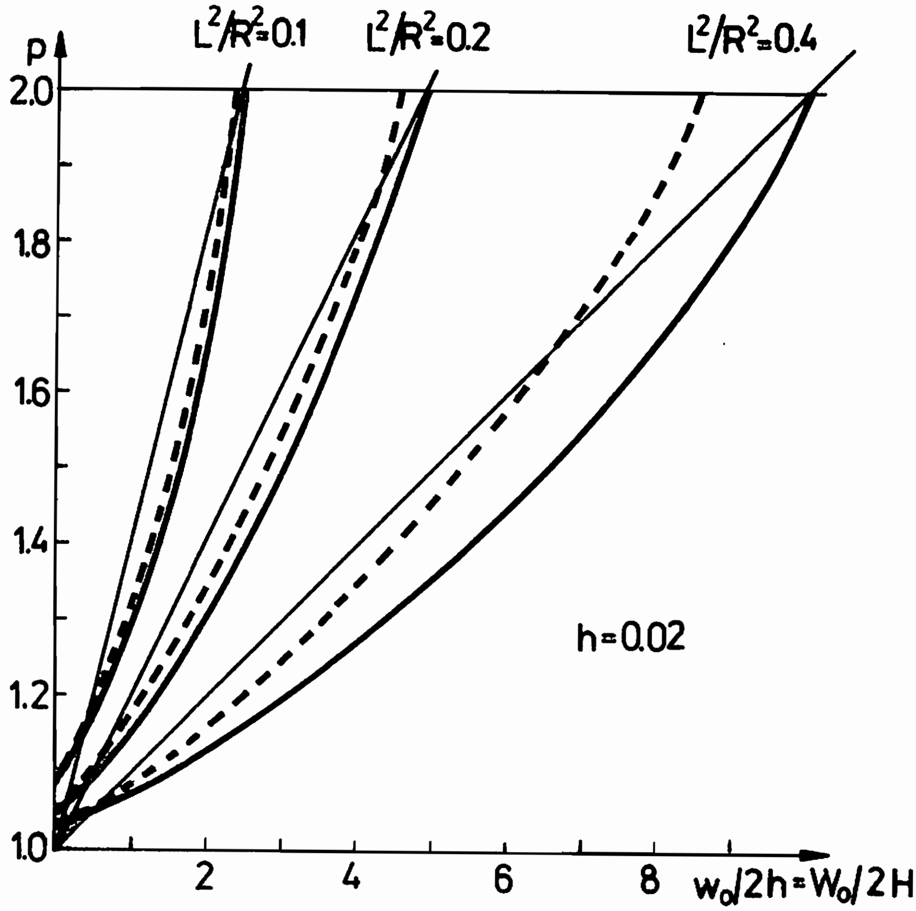

Fig. 10.4

10.2.4. Membrane solution

By substituting $m_x = 0$ and $n_x = n_\varphi = 1$ into the equilibrium equation $(10.9)_2$ we obtain the relation between the load and the second derivative of the deflection at the assumption of the membrane state,

$$w'' = \left(\frac{L}{R}\right)^2 (1 - p) \quad . \tag{10.43}$$

Integrating this equation twice with respect to x and making use of the boundary conditions $w(\pm 1) = 0$, $w'(0) = 0$ in order to determine the integration constants, we obtain the following relation between the load p and the deflection w_o in the membrane state

$$p = 1 + 2\left(\frac{R}{L}\right)^2 w_o \quad . \tag{10.44}$$

The membrane solutions (10.44) are shown in Fig. 10.4 by the straight lines.

10.3. Cylindrical shell, approximate elastic-plastic solution

A solution of the elastic-plastic shell problem, in general, involves several different regimes and the determination of the regime boundaries and the integration constants becomes very complicated. Therefore, the solution can be obtained using either numerical techniques or approximate approach.

Here, we shall limit ourselves to a discussion of an approximate solution of the elastic-plastic cylindrical shell problem, suggested by Paul and Hodge [45].

To this end we make use of the solution of the corresponding rigid-plastic problem.

The elastic relations between generalized stresses and generalized strains are easily computed from the Hooke's law, the results being:

$$\begin{aligned} \lambda_x^e &= \frac{\sigma_o}{E}(n_x - \nu n_\varphi) \quad , \\ \lambda_\varphi^e &= \frac{\sigma_o}{E}(n_\varphi - \nu n_x) \quad , \\ \kappa_x^e &= \frac{3\sigma_o}{4E}(1 - \nu^2) m_x \quad . \end{aligned} \tag{10.45}$$

The elastic changes in the shape of the shell are assumed to be sufficiently small so that the stress profile at the yield point load is the same as for the rigid plastic shell. Therefore, for the example of cylindrical shell considered in the Sec. 10.2, in view of (10.15), the plastic rate of curvature $\dot{\kappa}_x^p$ is equal zero. Thus, making use of the assumption of additive decomposition of the total strain into elastic and plastic parts and $(10.45)_3$ in $(10.7)_3$ we get

$$w'' = -2\alpha(\kappa_x^e + \kappa_x^p) = -2\alpha\kappa_x^e = -\frac{3\sigma_o}{2E}(1 - \nu^2)\alpha m_x . \tag{10.46}$$

Substitution of eq. (10.46) and $n_\varphi = +1$ into the equilibrium equation $(10.12)_1$ leads to

$$m''_x + \beta^2 m_x = 2\alpha(p - 1) \tag{10.47}$$

where

$$\beta^2 = \frac{3p}{2}\frac{\sigma_o}{E}(1 - \nu^2)\left(\frac{R}{L}\right)^2\alpha^2 \quad . \tag{10.48}$$

Integrating the eq. (10.47) with respect to x and making use of the boundary conditions (10.18) and (10.19), we obtain

$$m_x = \frac{2\alpha(1-p)}{\beta^2}\left(\frac{\cos\beta x}{\cos\beta} - 1\right) \quad . \tag{10.49}$$

Finally, the condition of plastic state at the centre of the shell

$$m_x(0) = -1 + n_x = -1 + \frac{p}{2} \tag{10.50}$$

enables to determine the elastic collapse load p_E as

$$p_E = 1 + \frac{\beta^2\cos\beta}{\beta^2\cos\beta + 4\alpha(1-\cos\beta)} \quad . \tag{10.51}$$

Equation (10.51) is an implicit representation since β, defined by eq. (10.48), depends on p.

Fig. 10.5, taken from the paper [45], shows the collapse load as a function of a parameter

$$c = \frac{1-\nu^2}{2}\left(\frac{R}{L}\right)^2\frac{\sigma_o}{E} \tag{10.52}$$

for various values of α.

The solution (10.51) is not an exact one, since the flow law is not ensured to be satisfied. However, as it was shown in [45], at least for short shells ($\alpha \le \frac{3\pi}{4}$) the load p_E determined by eq. (10.51) is the exact load for which the displacements become infinite.

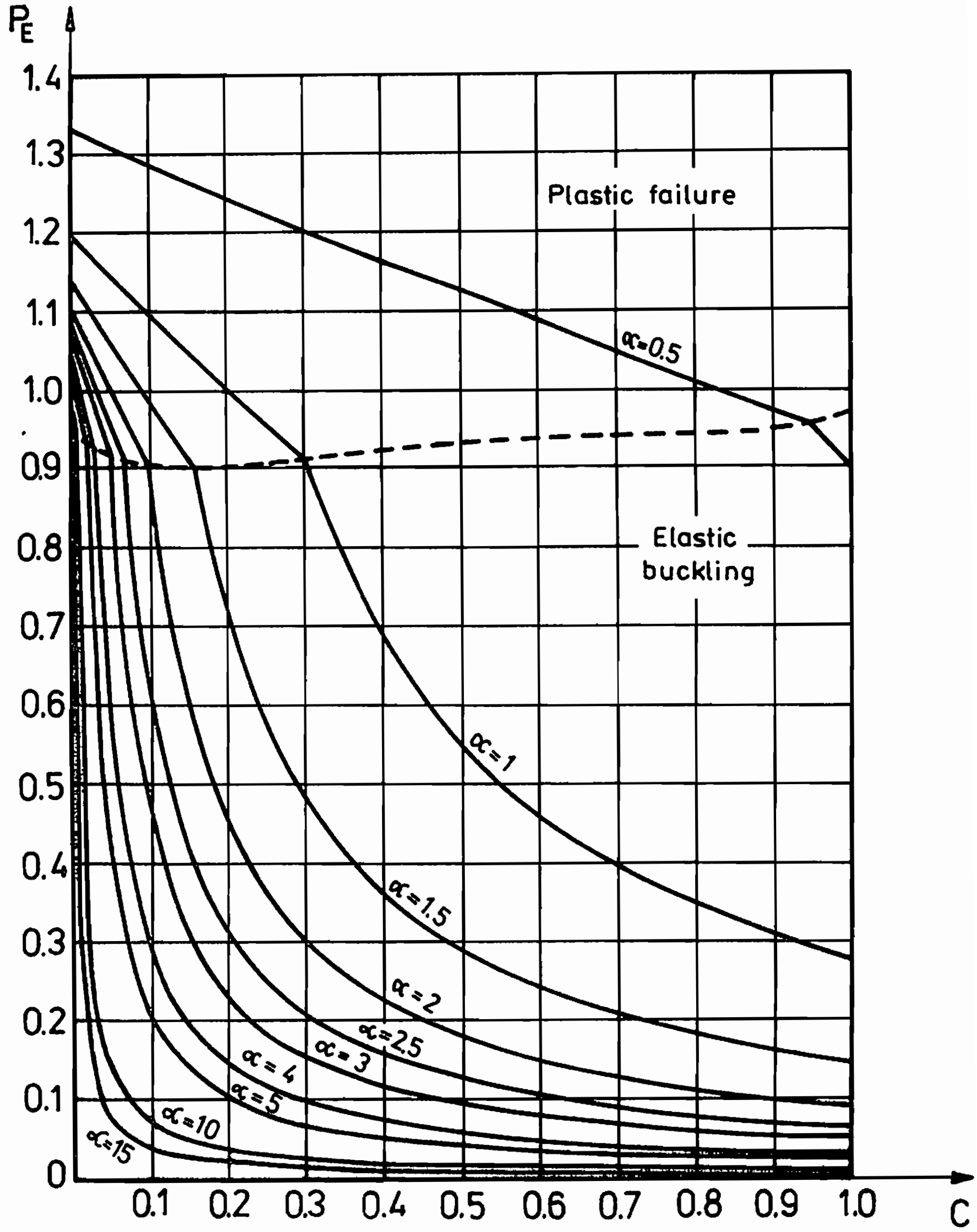
P_E
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
C
Plastic failure
Elastic buckling
α=0.5
α=1
α=1.5
α=2
α=2.5
α=3
α=4
α=5
α=10
α=15

Fig. 10.5

10.4. Shallow spherical shells, rigid-plastic solution

As the next example let us consider a shallow spherical cap loaded by a uniformly distributed internal dead load P (Fig. 10.6). The presented solution was obtained by M. Duszek [26]

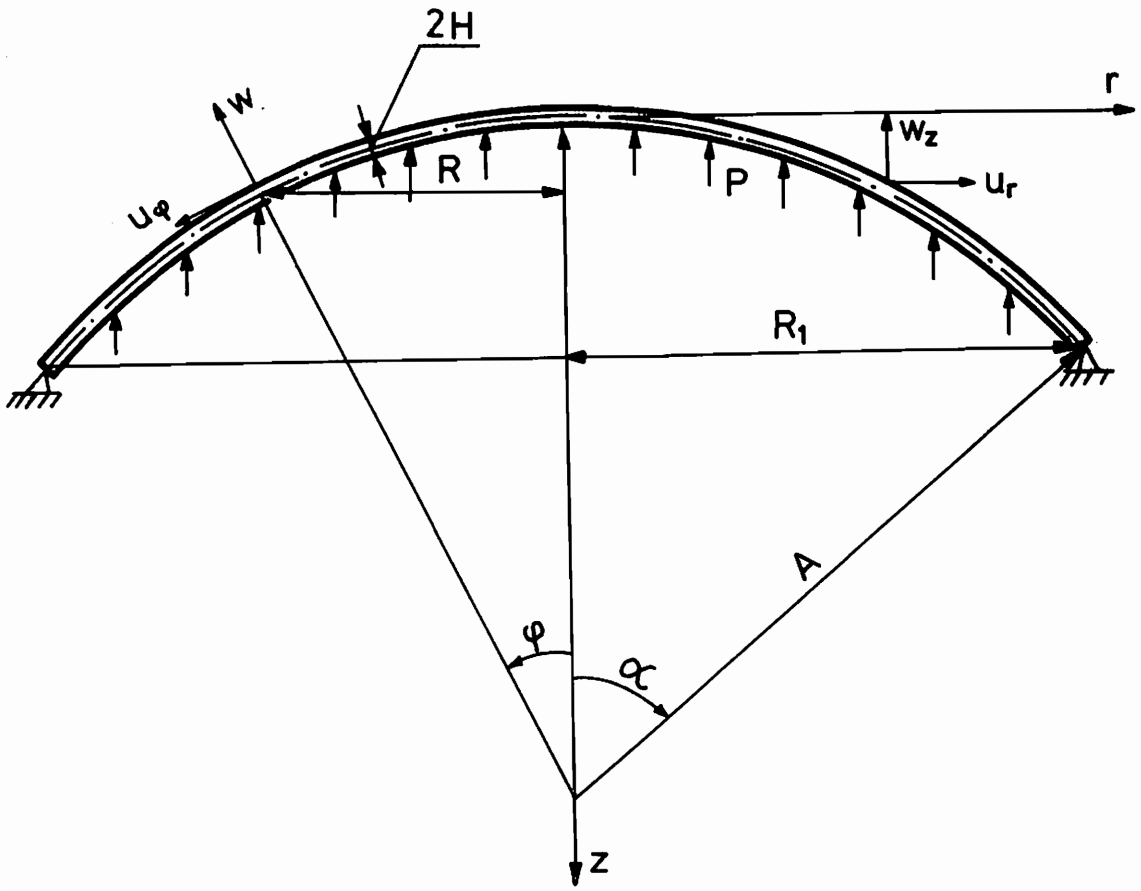

Fig. 10.6

The shell of radius A and constant thickness 2 H is made of rigid, perfectly plastic material, obeying the Tresca yield condition and the associated flow rule. The cross-section of the shell is assumed to be uniform. The problem is formulated in the spherical coordinate system (φ, θ, A). For further convenience we introduce the following dimemsionless quantities:

$$n_\varphi = \frac{N_\varphi^\varphi}{N_o} \quad , \quad n_\theta = \frac{N_\theta^\theta}{N_o} \quad , \quad m_\varphi = \frac{M_\varphi^\varphi}{M_o} \quad , \quad m_\theta = \frac{M_\theta^\theta}{M_o} \quad , \tag{10.53}$$

$$w = \frac{W}{A} \quad , \quad u_\varphi = \frac{V_\varphi}{A} \quad , \quad h = \frac{M_o}{AN_o} \quad , \quad c = \sin\alpha = \frac{R_1}{A} \quad , \tag{10.54}$$

$$r = \sin\varphi = \frac{R}{A} \quad , \quad \kappa_\varphi = HK_\varphi^\varphi \quad , \quad \kappa_\theta = HK_\theta^\theta \quad , \quad p = \frac{PA}{N_o} \tag{10.55}$$

whose meaning is evident from Fig. 10.6.

It will be shown that, within the framework of the theory of shallow shells at moderately large deflections, the transition from the yield point load state to purely membrane state is possible. Employing the geometrical relations listed in Table 5.1, case 1.b for the rotationally symmetric deformation of a spherical shell, we obtain

$$\begin{aligned} \lambda_\varphi &= u'_\varphi - w + \frac{1}{2}(w')^2 \quad , & \lambda_\theta &= u_\varphi \operatorname{ctg}\varphi - w \quad , \\ \kappa_\varphi &= -hw'' \quad , & \kappa_\theta &= -hw' \operatorname{ctg}\varphi \end{aligned} \tag{10.56}$$

where prime indicates differentiation with respect to the variable φ.

Transforming Eqs. (10.56), expressed in terms of the components w, u_φ of the displacement vector in the spherical coordinate system, into the cylindrical coordinate system (z, r, θ), and neglecting terms of the order of magnitude higher than ε^2, the strain-displacement relations take the form:

$$\begin{aligned} \lambda_\varphi &= u'_r + w'_z r + \frac{1}{2}(w'_z)^2 \quad , & \lambda_\theta &= \frac{u_r}{r} \quad , \\ \kappa_\varphi &= -hw''_z \quad , & \kappa_\theta &= -\frac{h}{r} w'_z \quad . \end{aligned} \tag{10.57}$$

Now prime indicates differentiation with respect to the radius. Since for shallow shells $\cos\varphi \approx 1$, we eventually have $\frac{d}{d\varphi} = \frac{d}{dr}$. The differentiation of (10.57) with respect to time provides the following expressions for the strain rates:

$$\dot{\lambda}_\varphi = \dot{u}_r' + \dot{w}_z' r + w_z' \dot{w}_z' \quad , \quad \dot{\lambda}_\theta = \frac{\dot{u}_r}{r} \quad ,$$
$$\dot{\kappa}_\varphi = -h\dot{w}_z'' \quad , \quad \dot{\kappa}_\theta = -\frac{h}{r}\dot{w}_z' \quad . \tag{10.58}$$

The equations of equilibrium (7.10), (7.11) obtained by means of the principle of virtual work take, for the considered case of shallow spherical cap, the form (cf. [26]),

$$(rn_\varphi)' - n_\theta = 0 \quad . \tag{10.59}$$

$$h[(rm_\varphi)' - m_\theta]' + [rn_\varphi(r + w_z')]' + rp = 0 \ . \tag{10.60}$$

The general form of the Tresca yield condition for uniform shells was derived (after Onat and Prager) in chapter 8. In terms of three parameters ξ_1, ξ_2, ξ_3 defined by Eqs (8.53) it is presented in Tables 8.2 and 8.3. The yield surface for $n_\theta = 0$ is visualized in Fig. 8.6.

Now, let us consider a spherical cap with simply supported edge restricted from any motion, and subjected to the dead load uniformly distributed over the plane (Fig. 10.6).

We assume that, at the yield point load, the membrane state is reached at the centre of the shell. Next, we suppose that the membrane zone propagates in the course of plastic deformations to cover a certain zone with radius z. Therefore, it is reasonable to suppose that (similarly to the cylindrical shell) two zones can be distinguished:

I. Central zone $0 \le r \le z$.

According to the introduced assumption the membrane state in the central zone is specified by

$$n_\varphi = n_\theta = 1 \quad , \quad m_\varphi = m_\theta = 0 \quad . \tag{10.61}$$

Substituting (10.61) into the equation of equilibrium (10.60), we obtain

$$(r^2 + rw_z')' + rp = 0 \ . \tag{10.62}$$

After integration and calculation of the integration constants from the boundary conditions: $w_z(0) = w_o$, $w_z'(0) = 0$; the following formula for the shell deflection in the central zone is obtained

$$w_z = -\frac{r^2}{2}\left(1 + \frac{p}{2}\right) + w_o \tag{10.63}$$

II. Boundary zone $z \le r \le c$.

The plastic stress state in the boundary zone is supposed to be represented by the yield hypersurface H_2^-(Fig.8.6) described, according to the Table 8.3, by the equation

$$n_\theta^2 - m_\theta = 1 \ . \tag{10.64}$$

The range of validity of this hypothesis will be discussed later. The associated strain-rate vector is

$$\dot\lambda_\varphi = 0 \quad , \quad \dot\lambda_\theta = 2\nu n_\theta \quad , \quad \dot\kappa_\varphi = 0 \quad , \quad \dot\kappa_\theta = -\nu \quad . \tag{10.65}$$

Substituting (10.58) into (10.65), we get

$$\begin{aligned} &\dot w_z'' = 0 \quad , \\ &\dot u_r' + \dot w_z' r + w_z' \dot w_z' = 0 \quad , \\ &\dot u_r = 2 n_\theta h \dot w_z' \quad . \end{aligned} \tag{10.66}$$

The relations (10.64) and (10.66) furnish a system of four equations in the four unknowns: w_z, u_r, n_θ and m_θ. This system has an elementary solution. Using the boundary conditions:

$$w_z = 0 \quad , \quad u_r = 0 \qquad \text{for } r = c \quad , \tag{10.67}$$

the continuity condition:

$$w_z'] = 0 \quad , \quad u_r] = 0 \qquad \text{for } r = z \quad , \tag{10.68}$$

and the initial conditions:

$$w_z = u_r = 0 \qquad \text{for } t = 0 \tag{10.69}$$

the solution is found to be

$$w_z = (1 + \frac{p}{2})(c - r)z , \tag{10.70}$$

$$u_r = \frac{rz}{2}(1 + \frac{p}{2})[r - z(1 + \frac{p}{2})] - \frac{cz}{2}(1 + \frac{p}{2})[c - z(1 + \frac{p}{2})] , \tag{10.71}$$

$$n_\theta = \frac{1}{2h}[\frac{1}{2}(c^2 - r^2) + z(r - c)(1 + \frac{p}{2})] , \tag{10.72}$$

$$m_\theta = \frac{1}{4h^2}[\frac{1}{2}(c^2 - r^2) + z(r - c)(1 + \frac{p}{2})]^2 - 1 . \tag{10.73}$$

The remaining functions n_φ and m_φ can be determined from the equilibrium equations. Therefore, substituting (10.70), (10.72) and (10.73) into (10.59) and (10.60), integrating and making use of continuity conditions

$$m_\varphi] = 0 \quad , \quad n_\varphi] = 0 \qquad \text{for } r = z , \tag{10.74}$$

we obtain:

$$n_\varphi = -\frac{1}{2h}[\frac{r^2}{6} - \frac{c^2}{2} + (1 + \frac{p}{2})(c - \frac{r}{2})z] + $$
$$+ \frac{z}{2hr}[\frac{z^2}{6} - \frac{c^2}{2} + (1 + \frac{p}{2})(c - \frac{z}{2})z] + \frac{z}{r} , \tag{10.75}$$

$$m_\varphi = \frac{7}{240h^2}r^4 - \frac{7}{48h^2}zr^3(1 + \frac{p}{2}) +$$
$$+ \{\frac{z^2}{6h^2}(1 + \frac{p}{2})^2 + \frac{1}{4h^2}[cz(1 + \frac{p}{2}) - \frac{c^2}{2}] - \frac{p}{6h}\}r^2 +$$
$$+ \{\frac{-z}{2h} - \frac{z}{4h^2}[\frac{z^2}{6} - \frac{c^2}{2} + (1 + \frac{p}{2})(c - \frac{z}{2})z] +$$
$$+ \frac{zc}{2h^2}(1 + \frac{p}{2})[-z(1 + \frac{p}{2}) + \frac{c}{2}]\}r +$$
$$+ \{\frac{z^2}{h}(1 + \frac{p}{2}) + \frac{z^2}{2h^2}(1 + \frac{p}{2})[\frac{z^2}{6} - \frac{c^2}{2} + (1 + \frac{p}{2})(c - \frac{z}{2})z] + \tag{10.76}$$

$$+ \frac{c^2}{4h^2}[- z(1 + \frac{p}{2}) + \frac{c}{2}]^2 - 1\} +$$

$$+ \{ \frac{z^4}{h^2} [\frac{1}{80} - \frac{1}{16} (1 + \frac{p}{2}) + \frac{1}{12} (1 + \frac{p}{2})^2] -$$

$$- \frac{z^2 p}{h} [\frac{1}{12} + \frac{1}{2p} (1 + \frac{p}{2})] - [\frac{c^2}{4h} - \frac{cz}{2h} (1 + \frac{p}{2})]^2 + 1\} \frac{z}{r} .$$

To find the complete solution, we have to determine the deflection in the centre of the shell w_o and the radius of the membrane zone z. Making use of the continuity and boundary conditions

$$w_z] = 0 \qquad \text{for } r = z , \tag{10.77}$$

$$m_\varphi = 0 \qquad \text{for } r = c \tag{10.78}$$

we finally obtain:

$$w_o = (1 + \frac{p}{2})(c - \frac{z}{2}) z , \tag{10.79}$$

where z being the function of p is determined by the following equation

$$\frac{7}{240h^2} c^4 - \frac{7}{48h^2} zc^3 (1 + \frac{p}{2}) +$$

$$+ \{ \frac{z^2}{6h^2} (1 + \frac{p}{2})^2 + \frac{1}{4h^2} [cz(1 + \frac{p}{2}) - \frac{c^2}{2}] - \frac{p}{6h} \} c^2 +$$

$$+ \{ - \frac{z}{2h} - \frac{z}{4h^2} [\frac{z^2}{6} - \frac{c^2}{2} + (1 + \frac{p}{2})(c - \frac{z}{2}) z] +$$

$$+ \frac{zc}{2h^2} (1 + \frac{p}{2}) [- z (1 + \frac{p}{2}) + \frac{c}{2}] \} c +$$

$$+ \{ \frac{z^2}{h} (1 + \frac{p}{2}) + \frac{z^2}{2h^2} (1 + \frac{p}{2}) [\frac{z^2}{6} - \frac{c^2}{2} + (1 + \frac{p}{2})(c - \frac{z}{2}) z] + \tag{10.80}$$

$$+ \frac{c^2}{4h} [-z(1 + \frac{p}{2}) + \frac{c}{2}]^2 - 1\} +$$

$$+ \{ \frac{z^4}{h^2} [\frac{1}{80} - \frac{1}{16} (1 + \frac{p}{2}) + \frac{1}{12} (1 + \frac{p}{2})^2] -$$

$$- \frac{z^2 p}{h} [\frac{1}{12} + \frac{1}{2p} (1 + \frac{p}{2})] - [\frac{c^2}{4h} - \frac{cz}{2h} (1 + \frac{p}{2})]^2 + 1 \} \frac{z}{c} = 0 .$$

An illustration of the above solution for two particular cases of the shells is shown in Fig. 10.7 by solid line.

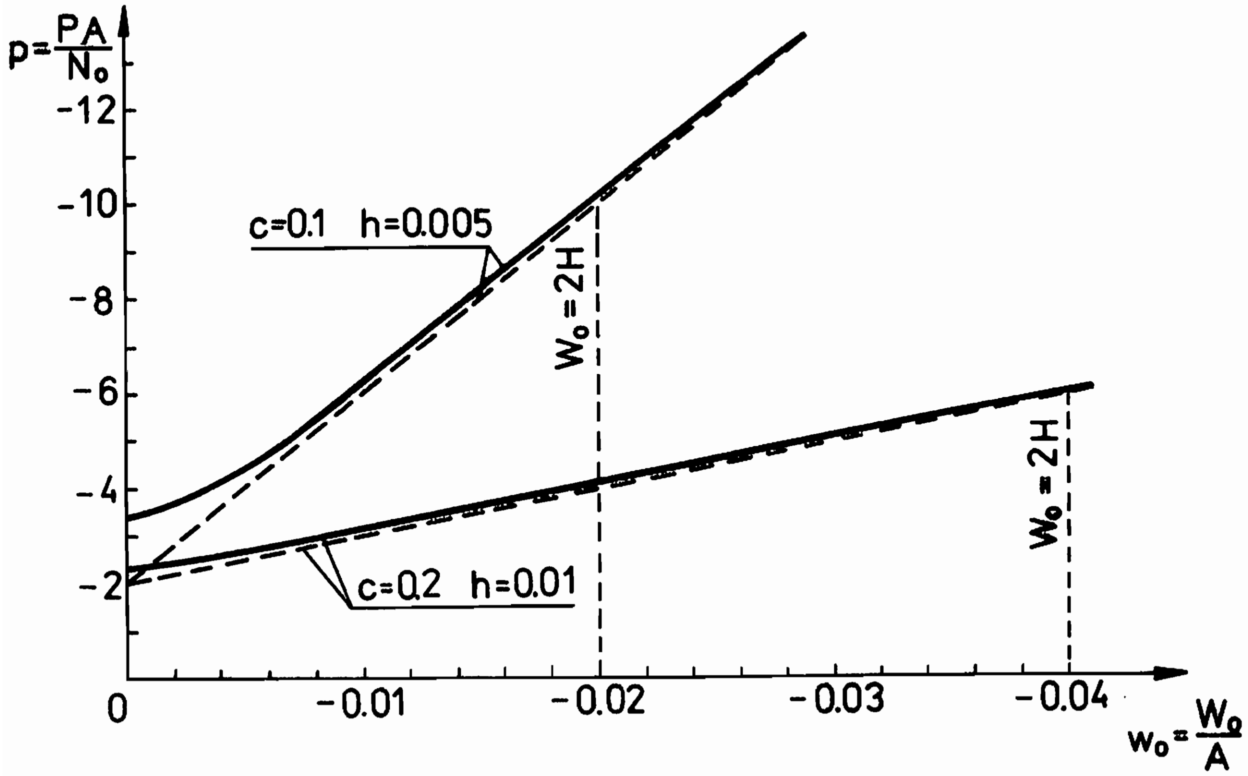

Fig. 10.7

A purely membrane solution, readily available from the equation of equilibrium (10.60), provides the following load-deflection relation

$$p_m = 2\left(\frac{2w_o}{c^2} - 1\right) \quad . \tag{10.81}$$

This solution is visualized in Fig. 10.7 as straight, broken lines. The results plotted clearly show a gradual transition from a bending to the membrane state where purely membrane state is reached asymptotically for $w \to \infty$. However, it should be remembered that the solution for shallow caps represented by the solid line is valid for deflections of the order of magnitude of shell thickness. For larger deflections, both equilibrium equations and geometrical relations become insufficiently accurate and therefore suitable strain-displacement relations from the Tabel 5.1 should be applied.

The propagation of the membrane zone as a function of the deflection $w_o = w_o(z)$, computed from (10.79) and (10.80), is shown in Fig. 10.8

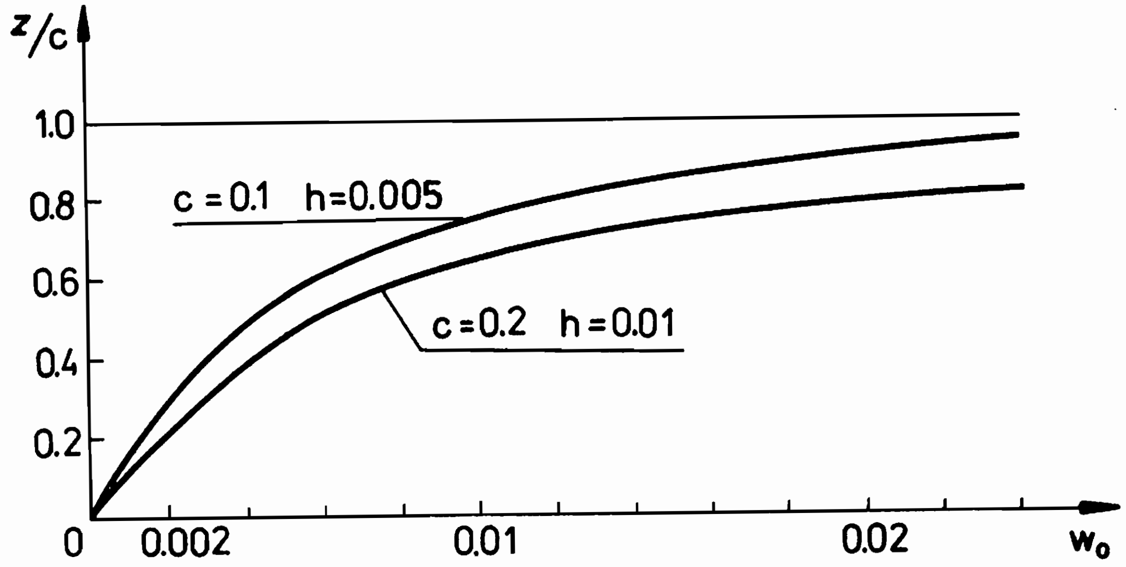

Fig. 10.8

The magnitude of yield-point load can be computed from (10.80) by putting $z = 0$. The result is found to be

$$p_o = -(6 \frac{h}{c^2} + 0{,}2 \frac{c^2}{h}). \tag{10.82}$$

Equation (10.72), (10.73), (10.75) and (10.76) show that the values of generalized stresses on the edge $r = c$ are $n_\theta = 0$, $m_\theta = -1$ $m_\varphi = 0$, $n_\varphi \neq 0$. This stress field exceeds the limit hypersurface of Fig. 8.6. Therefore, the obtained formulae provide only an upper estimate of the complete solution at the instant of yielding and an approximate solution for advanced plastic deformations. This solution can be, however, treated as an exact one for a suitably altered yield surface, for example a small outward parallel shifting of hypersurfaces H^+_{12} and F^+_2 satisfies this requirement.

The complete solution would involve additional zones corresponding to the relevant equations of the yield surfaces. However, such attempts cannot be more promising than the attempts to find the closed complete solution to the limit analysis problem, where the exact solution is still unknown.

Assuming $c \to 0$ and $Ac \to R_1$ a limit transition to the plate with the radius R_1 is obtained. In the notation used in [46]

$$q = \frac{PR_1}{6\sigma_o H^2} \quad , \quad \delta = \frac{W_o}{2H} \quad , \quad \xi = \frac{z}{c} \tag{10.83}$$

eqs. (10.79) and (10.80) in the limiting case yield (Fig. 10.9, solid line)

$$\begin{aligned} &\delta = \frac{3}{4} q\xi(1 - \frac{\xi}{2}) \quad , \\ &\frac{3}{4} q^2\xi^2(\xi^3 - 3\xi^2 + 3\xi - 1) + q(-2\xi^3 + 3\xi^2 - 1) - 1 + \xi = 0 \ . \end{aligned} \tag{10.84}$$

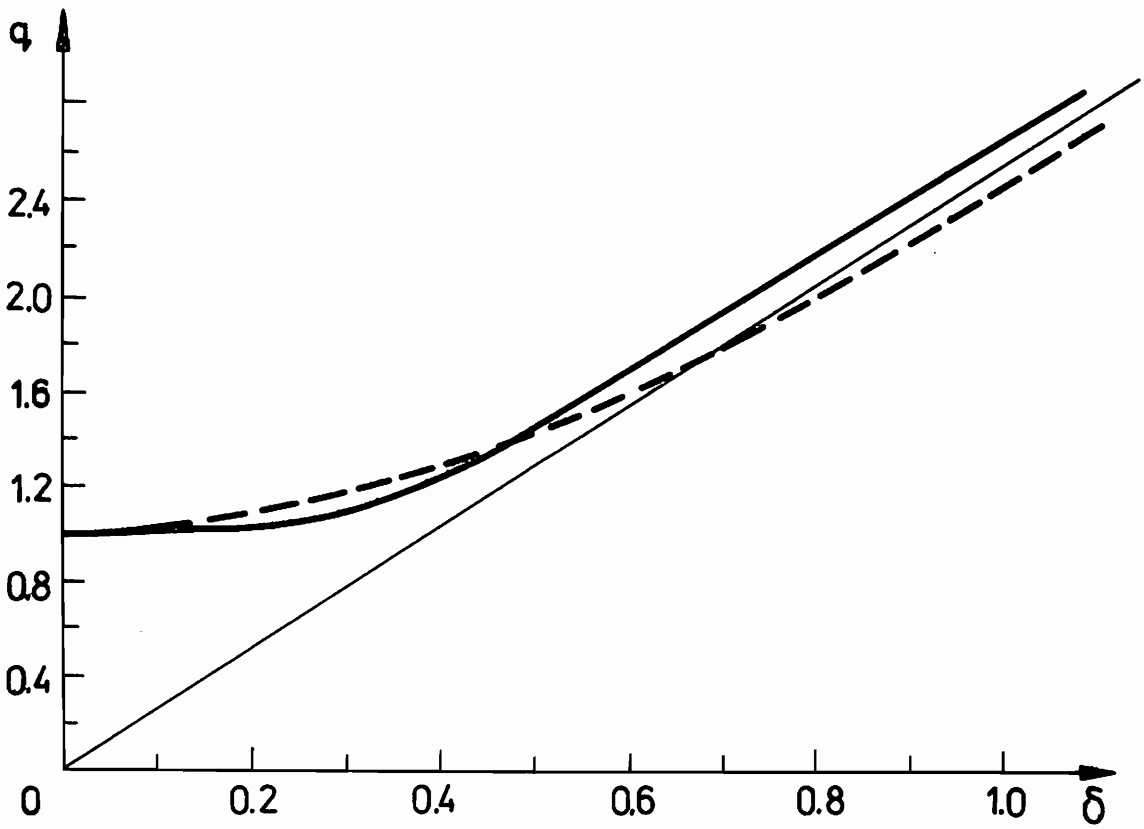

Fig. 10.9

The membrane solution for the plate can be obtained from $(10.84)_1$ by putting $\xi = 1$. The result is (Fig. 10.9, straight line).

$$q = \frac{8}{3}\,\delta\ . \tag{10.85}$$

The broken line in Fig. 10.9 corresponds to the solution for circular plates at moderately large deflections obtained by Ü. Lepik [46].

The curves in Fig. 10.8 indicate that the central membrane zone propagates very rapidly as deflections increase. The thinner the shell the faster the membrane zone propagates. At $h = 0.01$ and $h = 0.005$ for deflections equal to the shell thickness, the central zone extends over the major part of the shell, $z = 0.90c$ i. e. only a narrow region adjacent to the support is subjected to bending. The plot of the limit load versus the deflection w_o tends asymptotically to the membrane solution, as it was shown in Fig. 10.7. For the shell deflections equal to the wall thickness, the difference between the present solution p and the membrane solution p_m amounts approximately to one per cent as far as the value of the load-carrying capacity is concerned. Hence it can be concluded that the pure membrane state in the shell is practically reached at deflections of the order of the wall thickness.

10.5. Concluding remarks

The solutions presented in this chapter confirm the results known from literature [25], [39 - 41], [46 - 49], obtained when analysing plastic shells and plates at large deflections.

A general conclusion can now be drawn that the changes in geometry constitute an essential factor governing the response of structures within a plastic regime leading to the so called "geometrical strengthening". The opposite situation can also arise when the effect of "weakening" may be noticed [49]. Similarly as for plates [48] "unstable" load-deflection relations are then obtained.

In both cases, however, accounting for the deformed shape of a rigid-perfectly structure yields the load values differing significantly from the classical limit analysis solutions.

REFERENCES

[1] Von Kârmân, T., Festigkeitsprobleme im Maschinenbau, Encykl. d. Math. Wiss, vol. No. 4, 1910, pp. 331-385.

[2] Donnell, L. H., Stability of thin-walled tubes under torsion, NACA Rep. No. 479, 1933.

[3] Mushtari, Kh. M., Some generalizations of the theory of thin shells (in Russian), Izv. Fiz. Mat. Kazansk. Univ. 3, ser. 8, 1938.

[4] Vlasov, V. Z., General theory of shells and its technical application (in Russian) Gostekizdat, Moscow-Leningrad 1949.

[5] Hill, R., On constitute inequalities for simple materials, Mech. Phys. Solids, vol. 16, 1968, p. 229.

[6] Chien, W. Z., The intrinsic theory of thin shells and plates, Quart. App. Math., 1, 1944, pp. 297-327.

[7] Donnell, H. L., General thin shell displacement-strain relations, Proc. 4th U.S. Nat. Congress Appl. Mech., 1962.

[8] Reissner, E., Rotationally symmetric problems in the theory of thin plastic shells, Proc. 3rd U.S. Nat. Congress Appl. Mech., 1958.

[9] Koiter, W. T., On the nonlinear theory of thin elastic shells, Proc. Kon. Ned. Ak. Wet., Ser. B, 69, 1, 1966, pp. 1-54.

[10] John, F., Estimates for the derivatives of the stresses in a thin shell and interior shell equations, Comm. Pure Appl. Math., 18, 1965, pp. 235 - 267.

[11] Naghdi, P. M., Foundations of elastic shell theory, in: Progress in Solid Mech., 4, North Holland, Amsterdam, 1963.

[12] Sanders, J. L., An improved first-approximation theory for thin shells, NASA Report 24, 1959.

[13] Sanders, J. L., Non-linear theories for thin shells, Quart. Appl. Math., 21, 1963, pp. 21 - 36.

[14] Danielson, D. A., Simplified intrinsic equations for arbitrary elastic shells, Int. J. Eng. Sci., 8, 1970, pp. 151 - 259.

[15] Simmonds, J. G., Danielson, D. A., Non-linear shell theory with a finite rotation vector, Proc. Kon. Ned. Ak. Wet., Ser. B, 73, 1970, pp. 460 - 478.

[16] Pietraszkiewicz, W., Finite rotations and Lagrangean description in the non-linear theory of shells, Polish Scientific Publishers (PWN), Warszawa-Poznań 1979.

[17] Ilyushin, A. A., Plasticity (in Russian) Gostekizdat, Moscow 1948.

[18] Sawczuk, A., Rychlewski, J., On yield surface for plastic shells, Arch. Mech. Stos., 12, 1960, pp. 29 - 53.

[19] Hodge, P. G., Limit analysis of rotationally symmetric plates and shells, Prentice Hall, Englewood Cliffs., N. J., 1963.

[20] Sawczuk, A., Yield surfaces, Technical Note No. 1, University of Waterloo, March 1971.

[21] Huber, M. T., Specific work of deformation as a measure of the level of shearing (in Polish) Czasopismo Techniszne, Lemberg 1904.

[22] Von Mises, R., Mechanik der festen Körper im plastisch deformablen Zustand, Göttinger Nachr. Math.-Phys. K. 1913, p. 582.

[23] Duszek, M. K., Stability analysis of rigid plastic structures at the yield-point load, Proc. IUTAM symp., Cambridge 1974, Springer-Verlag Berlin 1974, pp. 106 - 116.

[24] Duszek, M. K., Problems of geometrically non-linear theory of plasticity, Mitteilungen des Instituts für Mechanik, Ruhr-Universität Bochum, No. 21, Juni 1980.

[25] Duszek, M. K., Plastic analysis of cylindrical shells subjected to large deflections, Arch. Mech. Stos., 5, 18, 1966, pp. 599 - 614.

[26] Duszek, M. K., Plastic analysis of shallow spherical shells at moderately large deflections, Proc. IUTAM Symp. Copenhagen 1967; Springer-Verlag, Berlin 1969, pp. 374 - 388.

[27] Ilyushin, A. A., Finite relations between forces and moments and their relation to the deformation of shells (in Russian), Prikl. Mat. Mekh., 9, 1945, pp. 101 - 110.

[28] Hodge, P. G., The Mises yield condition for rotationally symmetric shells, Quart. Appl. Math., 18, 1961, pp. 305 - 311.

[29] Sawczuk, A., On plastic shells at large strains and displacements, CISM Course, Udine, October 1979.

[30] Drucker, D. C., Limit analysis of cylindrical shells under axially-symmetric loading, Proc. 1-st Midwest Conf. Solid Mech., (Urbana 1953), ASME, New York, pp. 158 - 163.

[31] Onat, E. T., Prager, W., Limit analysis of shells of Revolution, Proc. Ned. Akad. Wetensch., Ser. B, 57, 534 - 548.

[32] Hodge, P. G., Plastic analysis of structures, Mc-Graw Hill Book, Co., Inc., New York 1959.

[33] Gwozdew, A. A., The determination of the value of the collapse load for statically indeterminate system undergoing plastic deformation, "Proceedings of the Conference on Plastic Deformation", Akademia Nauk SSSR, Moscow-Leningrad, 1938, pp. 19 - 33.

[34] Greenberg, H. J., Prager, W., Limit design of beams and frames, Proc. ASCE 77 (Sep. 59), 1951.

[35] Drucker, D. C., Greenberg, H. J., Prager W., The safety factor of an elastic-plastic body in plane strain, J. Appl. Mech., 18, 1951, pp. 371 - 378.

[36] Hill, R., On the state of stress in a plastic-rigid body at the yield point, Phil. Mag., 42, 7, 1951, pp. 686-875.

[37] Prager, W., Hodge, P. G., Theory of Perfectly Plastic Solids, John Willey and Sons, Inc., New York 1951.

[38] Prager, W., An Introduction to Plasticity, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959.

[39] Duszek, M. K., Sawczuk, A., Load-deflection relations for rigid-plastic cylindrical shells beyond the incipient collapse load, Int. J. Mech. Sci., 12, 1970, pp. 839 - 848.

[40] Dudyk, J., Duszek, M. K., Powłoki stożkowe z materiału sztywno-idealnie plastycznego poddane dużym ugięciom, Prace I.P.P.T., 21, 1974.

[41] Mitow, M., Duszek, M., Analiza powłok sztywno-plastycznych w zakresie dużych ugięć, Rozprawy Inż., 26, 4, 1978, pp. 575 - 594.

[42] Grigoriew, A.S., Equilibrium of momentless, rotationally symmetric shells at large deformations (in Russian), Prikl. Mat. Miech., 6, 25, 1961.

[43] Orkisz, J., Finite deformation of a circularly symmetric shell under membrane state of stress in some nonlinear cases, Bull. Acad. Polon. Sci., ser. sci. Techn., 1, 15, 1967, pp. 31 - 40.

[44] Orkisz, J., Skończone odkształcenia wiotkich osiowo-symetrycznych pwołok w stanie błonowym w świetle teorii płynięcia plastycznego, Mech. Teoret. i. Stos., 4, 5, 1967.

[45] Paul, B., Hodge, P. G., Carrying capacity of elastic-plastic shells under hydrostatic pressure, Proc. 3-rd U.S. Nat. Congr. Appl. Mech. (Providence, 1958), 1958, pp. 631 - 640.

[46] Lepik, Ü., Plastic flow of thin circular plates of a rigid-plastic material (in Russian), Bull. Acad. Sci. USSR, Mech. and Mech. Eng., 2, 1960, pp. 78 - 87.

[47] Lepik, Ü., Large deflections of rigid-plastic cylindrical shells under axial tension and external pressure, Nucl. Eng. Design, 4, 1966, pp. 29 - 38.

[48] Lepik, Ü., A contribution ot the axial-symmetric bending of thin circular rigid-plastic plates (in Russian), Mech. of Solids, 4, 1966, pp. 104 - 110.

[49] Shablij, O. N., Large deflections of a rigid-plastic shallow spherical shell (in Russian). Notes of the summer-school on "Physically and geometrically nonlinear problems of the theory of plates and shells", Tartu, 1966.

[50] Duszek, M. K., A systematic study of kinematics of shells at large strains and displacements, Bull. of Polish Acad. of Sci., 26, 1, 1978, pp. 39 - 47.

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