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Approximations and Error Estimates in Eigenvalue Problems of Elastic Systems with Application to Eigenvibrations of Orthotropic Plates

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Zusammenfassung

Die Lösung des Eigenschwingungsproblems linear-elastischer Kontinua führt zu einer unendlichen Folge von Eigenwerten mit zugeordneten Eigenschwingungszuständen. Die effektivste Methode zur Berechnung von Eigenwerten ist das klassische Rayleigh-Ritz Verfahren bzw. die hiervon ausgehenden geometrisch-kompatiblen finiten Verfahren, die obere Schranken für die exakten Eigenwerte liefern. Wie in der vorliegenden Arbeit gezeigt wird, lassen sich gleichzeitig untere Eigenwertschranken mit Hilfe oberer Schranken für die Invarianten des Greenschen Integraloperators berechnen, womit man eine befriedigende Aussage über die Qualität der Eigenwertapproximation erhält.

Die geometrisch-kompatiblen Verfahren liefern den Eigenwerten zugeordnete Näherungen für die entsprechenden Eigenzustände. Hierfür werden pauschale Fehlerschranken mit Hilfe oberer und unterer Eigenwertschranken berechnet. Darüberhinaus wird gezeigt, daß man mit Hilfe entsprechender Greenscher Zustände punktweise Schranken für beliebige Feldgrößen eines Eigenzustandes erhalten kann. Schließlich wird die numerische Realisierung der angegebenen Verfahren für orthotrope Platten untersucht.

Summary

The solution of the eigenvibration problem of linear-elastic continua leads to an infinite series of eigenvalues with associated eigenstates. The moste efficient method for the calculation of eigenvalues is the classical Rayleigh-Ritz method or equivalent geometrical-compatible finite procedures, which yield upper bounds for the exact eigenvalues. It is shown in this paper that lower eigenvalue bounds can be obtained by calculating upper bounds for the invariants of Green's integral operator, leading to a satisfying statement about the quality of the eigenvalue approximation.

Together with the eigenvalues the geometrical-compatible procedures yield approximations to the eigenstates, for which global error estimates are obtained by using upper and lower eigenvalue bounds. Furthermore it is shown that with appropriate Green's states also pointwise error bounds can be derived for arbitrary field quantities of the eigenstates. Finally the numerical application of the given methods is considered for orthotropic plate problems.

CONTENTS

1. INTRODUCTION

The free vibrations of linear-elastic systems can be described by a classical eigenvalue problem with a symmetric linear differential operator A. The most common method of approximating the fundamental frequencies of the elastic system and eigenvalues of A, respectively, is the classical Rayleigh-Ritz method, which allows a discrete representation and finite approximation procedure for continuous structural systems. Computing the eigenvalues as minima of the Rayleigh quotient in a vector space, the reduction of the eigenvalue problem on a space of large or infinite number of dimensions to an eigenvalue problem on a space of relatively few dimensions leads to an upper bound approximation. In spite of an easy application and a minimal amount of computational effort the Rayleigh-Ritz upper bounds need the complement of an adequate lower bound approximation to calculate the maximum possible error.

For the lower bound a number of approaches has been proposed in the literature, but missing a satisfying efficiency they have not gained a popularity in engineering comparable to the Rayleigh-Ritz procedure. Courant [1920] first remarked that a weakening of prescribed conditions leads to lower bounds for the exact eigenvalues. Weinstein, as summarized by Gould [1957] or Weinstein and Stenger [1972], introduced the method of "intermediate problems", which has been extended by Aronszajn [1943] [1951]. The theory consists in constructing a "base problem" with relaxed boundary conditions, for which an exact solution is available, followed by a sequence of "intermediate problems" yielding a lower bound approximation for the eigenvalues. Bazley [1961] and Bazley and Fox [1961] [1962 a] presented a procedure related to that of Weinstein and Aronszajn using the decomposition of the strain energy into a first part, corresponding to an exactly resolvable eigenvalue problem, and a second positive part. The lower bounds are obtained from the roots of finite determinental equations. Starting with rough lower bounds obtained by solving a base problem, Chang and Craig [1973] considered the improvement of such lower bounds by applying an operator decomposition suggested by Kato [1953].

As pointed out by Trefftz [1933] lower bounds can be calculated by using Rayleigh-Ritz upper bounds, if the first invariant of the analogous Green's integral operator is known. Fichera [1966] generalized this method by introducing the orthogonal invariants of a completely continuous operator. The crucial point lies in the fact that orthogonal invariants cannot be computed in general, because the kernel of the Green's integral operator is known only for special problems. This difficulty can be overcome by computing upper bounds for appropriate invariants, which had been considered by Fichera for homogeneous geometric boundary conditions and by Stumpf [1970][1972 a] and Rieder [1972] for arbitrary homogeneous boundary conditions.

In contrary to a vast literature on eigenvalue bounding methods error estimates for eigenvectors are subject only of a relatively few number of papers, among which are mentioned here the papers of Weinberger [1960], Bazley and Fox [1962], Bramble and Payne [1963], Moler and Payne [1968] and Stumpf [1970][1972 a,b]. Using Rayleigh-Ritz approximations global error bounds are given by Weinberger [1960] for the eigenvectors of a symmetric linear operator and also pointwise error bounds for the case of a special second order operator in two dimensions. Bazley and Fox [1962] estimate the difference in norm between an eigenfunction and a given vector in the domain of existence of the operator A, while Bramble and Payne [1963] consider bounds applicable in forced vibration problems of Dirichlet type. Moler and Payne [1968] derive mean square bounds for the eigenfunctions of self-adjoint elliptic differential operators and pointwise error bounds for the special case of a second order partial differential operator with homogeneous geometric boundary conditions.

The solution of the eigenvalue problem for continuous systems leads to an infinite series of eigenvalues with corresponding elastic eigenstates. It is shown by Rieder [1968][1972] and Stumpf [1970][1972 a,b,c] that pointwise error bounds for arbitrary field quantities of an elastic eigenstate can be obtained by using appropriate Green's states, if upper and lower bounds of the corresponding eigenvalues are known.

In this paper error estimates in eigenvalue problems of continuum mechanics and the numerical application to eigenvibrations of orthotropic plates are considered. Upper and lower bounds for the first ten eigenfrequencies, global error estimates for the first elastic eigenstate and pointwise bounds for the corresponding displacement field and also for the corresponding stress couples are given. It should be mentioned here that eigenvalue bounds for the natural frequencies of clamped rectangular orthotropic plates had been computed recently by

Marangoni, Cook and Basavanhally [1978], who used the Rayleigh-Ritz and Bazley-Fox techniques.

2. EIGENVALUE PROBLEM OF LINEAR CONTINUUM MECHANICS

Let Ω be the domain of a linear elastic body with volume V and surface S consisting of a part S_1 with given geometric boundary conditions and a part S_2 with given static boundary conditions. If geometric and static quantities are prescribed on the same part $S_1 \cap S_2$ of the surface, they must be mutually complementary to each other. Introducing a rectangular Cartesian coordinate system the deformation of the elastic body can be described by a displacement vector u with components $u_i\,(i=1,2,3)$ joining the position x of a material point in the reference configuration to its position in the deformed configuration.

The governing equations of the boundary value problem of linear elasticity can be given in the following form:

$$\left.\begin{array}{ll}\text{strain tensor} & \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \text{constitutive equations} & \sigma_{ij} = c_{ijkl}\varepsilon_{kl} \\ & \varepsilon_{ij} = s_{ijkl}\sigma_{kl} \\ \text{equilibrium equations} & \sigma_{ij,j} + \rho b_i = 0\end{array}\right\} \qquad (2.1)$$

$$\left.\begin{array}{lll}\text{geometric boundary conditions} & u_i(x) = u_i^*(x) & \forall x \in S_1 \\ \text{static boundary conditions} & \sigma_{ij}n_j(x) = p_i^*(x) & \forall x \in S_2\end{array}\right\} \qquad (2.2)$$

In Eqs. (2.1) c_{ijkl} is the fourth-order tensor of elastic moduli and s_{ijkl} the tensor of elastic constants. They are inverse to each other and satisfy the well-known symmetry conditions. With ρ we denote the mass density, with b_i the vector of body forces and with p_i the vector of boundary forces. Asterisks indicate given values on the boundary. Here and henceforth the usual summation convention will be used.

From Eqs. (2.1)(2.2) follow the relations defining the free vibrations of a linear elastic system, if in Eq. $(2.1)_4$ the body forces are replaced by inertia forces due to harmonic motions

$$\sigma_{ij,j} + \rho\omega^2 u_i = 0 \; , \tag{2.3}$$

and if the boundary conditions (2.2) are assumed to be homogeneous:

$$\begin{aligned} u_i(x) &= 0 \qquad \forall x \in S_1 \\ \sigma_{ij} n_j(x) &= 0 \qquad \forall x \in S_2 \end{aligned} \quad . \tag{2.4}$$

In Eq. (2.3) ω is the circular natural frequency.

Introducing Eqs. $(2.1)_1$ $(2.1.)_2$ into Eqs. (2.3) (2.4) the eigenvibrations of continuous elastic systems are given by:

$$\begin{aligned} &Au - \lambda u = 0 \\ &u \in \mathcal{D}_A \subset L^2(\Omega) \end{aligned} \tag{2.5}$$

with the differential operator

$$Au = -(c_{ijkl}\varepsilon_{kl}(u))_{,j} e_i \quad , \tag{2.6}$$

where e_i is the unit vector in the direction of x_i.

The displacement vectors u are considered as elements of the vector space $L^2(\Omega)$ of square integrable functions provided with a scalar product $<.,.>$ and a norm $\|\,.\,\|_{L^2}$:

$$\forall u,v \in L^2(\Omega) \longmapsto <u,v> = \int_\Omega u_i v_i dV \quad ; \quad \|u\|_{L^2} = [\int_\Omega u_i u_i dV]^{1/2} \quad . \tag{2.7}$$

The operator A is defined in the domain of existence $\mathcal{D}_A \subset L^2(\Omega)$ with elements $u \in \mathcal{D}_A$ sufficiently regular and satisfying the homegeneous boundary conditions (2.4). It can be shown easily that A is symmetric and positive:

$$<Au,v> = -\int_\Omega (c_{ijkl}\varepsilon_{kl}(u))_{,j} v_i dV = \int_\Omega \varepsilon_{ij}(u) c_{ijkl}\varepsilon_{kl}(v) dV = <u,Av> \tag{2.8}$$

$$<Au,u> = \int_\Omega \varepsilon_{ij}(u) c_{ijkl}\varepsilon_{kl}(u) dV \geq 0 \quad . \tag{2.9}$$

Furthermore A is positive definite in $\mathcal{D}_A$ with well-chosen homogeneous boundary conditions (2.4).

Defining a new scalar product on $\mathcal{D}_A$ for all $u,v \in \mathcal{D}_A$ by setting

$$\langle Au,v\rangle = \{u,v\} \tag{2.10}$$

with the norm:

$$\|u\| = \sqrt{\langle Au\ u\rangle}\ , \tag{2.11}$$

it becomes a Hilbert space H by completion, where the elements $u \in H$ have to satisfy the geometric boundary conditions.

The solution of the eigenvalue problem (2.5) is characterized by an infinite series of eigenvalues λ_n with corresponding eigenfunctions u_n

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots; \qquad u_1\ ;\ u_2\ \dots\ ;\ u_n\ ;\ \dots\ , \tag{2.12}$$

where each eigenvalue is repeated as many times as its multiplicity. We may assume that the eigenfunctions are an orthonormal system:

$$\langle u_n,u_m\rangle = \delta_{nm} \qquad \langle Au_n,u_m\rangle = \lambda_n\delta_{nm}\quad . \tag{2.13}$$

The eigenvalues λ_n can be obtained by minimizing the Rayleigh-quotient

$$\lambda_n = \min_{u\in H} \frac{\{u,u\}}{\langle u,u\rangle} \quad \text{for } \langle u,u_m\rangle = 0 \quad ; \quad m = 1,2,\dots, (n-1)\quad , \tag{2.14}$$

which is equivalent to the well-known minimax principle:

$$\lambda_n = \max_{V_{n-1}\subset H}\ \left[\ \min_{u\in H-V_{n-1}} \frac{\{u,u\}}{\langle u,u\rangle}\ \right]\quad , \tag{2.15}$$

where $V_{n-1} \subset H$ is a linear variety spanned by n - 1 vectors $v_1,\dots,$ v_{n-1} $(n \geq 1)$ of the Hilbert space H.

3. RAYLEIGH-RITZ UPPER BOUNDS FOR EIGENFREQUENCIES

According to the Rayleigh-Ritz approximation we have to choose a set of ν linearly independent elements φ_1 ; φ_2 ; ... ; φ_ν complete in H. They span a ν-dimensional subspace $V_\nu \subset H$ such that every vector $u \in V_n$ can be represented in the form

$$u = \sum_{\beta=1}^{\nu} a_\beta \varphi_\beta \tag{3.1}$$

With Eq. (3.1) the minimizing process of Eq. (2.14) and Eq. (2.15), respectively, on a space of infinite dimensions H can be reduced on a space of finite dimensions V_ν. The roots of the determinantal equation

$$\det\,[\{\varphi_\alpha,\varphi_\beta\} - L^{(\nu)}\langle\varphi_\alpha,\varphi_\beta\rangle] = 0 \qquad (\alpha,\beta = 1,\ldots,\nu) \tag{3.2}$$

yield a sequence of eigenvalue approximations

$$L_1^{(\nu)} \leq L_2^{(\nu)} \leq \ldots \leq L_\nu^{(\nu)} \quad , \tag{3.3}$$

which are upper bounds to the exact eigenvalues λ_n:

$$\lambda_n \leq L_n^{(\nu+1)} \leq L_n^{(\nu)} \qquad \lim_{\nu\to\infty} L_n^{(\nu)} = \lambda_n \quad . \tag{3.4}$$

Associated with the Rayleigh-Ritz eigenvalues (3.3) are eigenvectors

$$v_n^{(\nu)} = \sum_{\beta=1}^{\nu} a_\beta^{(\nu)} \varphi_\beta \ , \tag{3.5}$$

which are approximations of the exact eigenvectors u_n. The coefficients $a_\beta^{(\nu)}$ of Eq. (3.5) are defined by

$$\sum_{\beta=1}^{\nu} a_\beta^{(\nu)} [\{\varphi_\alpha,\varphi_\beta\} - L_n^{(\nu)}\langle\varphi_\alpha,\varphi_\beta\rangle] = 0 \qquad (\alpha = 1,\ldots,\nu)$$

$$\sum_{\alpha,\beta=1}^{\nu} \langle\varphi_\alpha,\varphi_\beta\rangle a_\alpha^{(\nu)} a_\beta^{(\nu)} = 1 \quad . \tag{3.6}$$

Then

$$\langle v_n^{(\nu)}, v_m^{(\nu)} \rangle = \delta_{nm} \; ; \; \{v_n^{(\nu)}, v_m^{(\nu)}\} = L_n^{(\nu)} \delta_{nm} \quad (n,m = 1,\ldots,\nu) \; . \qquad (3.7)$$

4. ESTIMATES FOR EIGENVECTORS

To determine the degree of approximation of the Rayleigh-Ritz eigenvectors v_n to the exact eigenvectors u_n, an upper bound of the difference in norm $\langle v_n - u_n, v_n - u_n \rangle$ must be known. Such error bounds have been considered by Weinberger [1960], Bazley and Fox [1962 b], Rieder [1968] and Stumpf [1970][1972 b]. For the numerical computation of orthotropic plate problems a formula of Weinberger will be used in this paper.

It is assumed that the eigenvalues are seperated $\lambda_{n-1} < \lambda_n < \lambda_{n+1}$ and that sufficiently good upper bounds L_{n-1}, L_n and lower bounds l_n, l_{n+1} are known such that

$$L_{n-1} < l_n < L_n < l_{n+1} \quad . \qquad (4.1)$$

To calculate an upper bound for

$$\langle v_n - u_n, v_n - u_n \rangle = 2(1 - a_n^n) \qquad (4.2)$$

or a lower bound for a_n^n the following theorem is proposed by Weinberger [1960].

THEOREM:

Let $L_1 \leq \ldots \leq L_\nu$ be the Rayleigh-Ritz upper bounds for the first ν of the eigenvalues $\lambda_1 < \lambda_2 < \ldots$ of a symmetric linear operator A. Let $l_1 < \ldots < l_\nu$ be lower bounds for the first ν eigenvalues of A such that relation (4.1) is valid.

Define the numbers

$$i_1 < i_2 \ldots < i_{N+1} \; ; \; \beta_1 < \beta_2 \ldots < \beta_N \quad \text{by}$$

$$i_1 = 1 \; ; \; i_2 = \min \{i | l_i > L_1\} \; ; \quad \beta_1 = \max \{\beta | L_\beta \leq l_{i_2}\} \; ;$$

$$i_{\mu+1} = \min \{i \; \exists \; \beta \ni l_{i_\mu} < L_\beta < l_i\} \; ; \; \beta_\mu = \max \{\beta | L_\beta \leq l_{i_{\mu+1}}\} \; .$$

Then if v_n is the normalized Rayleigh-Ritz eigenvector corresponding to the bound L_n and u_n is the normalized eigenvector of A corresponding to the eigenvalue λ_n

$$(a_n^n)^2 \geq \left\{1 - \frac{L_n - l_n}{l_{i_{N+1}} - l_n}\right\} \prod_{\substack{\nu=1 \\ \beta_\nu \neq n}}^{N} \left\{1 - \frac{(L_n - l_n)(L_{\beta_\nu} - l_{i_\nu})}{(L_n - L_{\beta_\nu})(l_n - l_{i_\nu})}\right\} . \tag{4.3}$$

In addition we report a similar result given by Bazley, Fox [1962 b]. Define the numbers γ_i by

$$\gamma_i = \begin{cases} l_i - \langle Av_n, v_n\rangle & , \quad \langle Av_n, v_n\rangle < l_i \\ 0 & , \quad l_i \leq \langle Av_n, v_n\rangle \leq L_i \\ \langle Av_n, v_n\rangle - L_i & , \quad \langle Av_n, v_n\rangle > L_i \end{cases} \tag{4.4}$$

Then

$$(a_n^n)^2 \geq \frac{l_{N+1} - \langle Av_n, v_n\rangle}{l_{N+1} - l_n} - \frac{\langle Av_n, Av_n\rangle - \langle Av_n, v_n\rangle^2}{l_{N+1} - l_n} \sum_{\substack{i=1 \\ i \neq n}}^{N} \frac{l_{N+1} - l_i}{\langle Av_n, Av_n\rangle - \langle Av_n, v_n\rangle^2 + \gamma_i^2} \tag{4.5}$$

Compared with Eq. (4.3) the formula of Bazley and Fox requires the computation of the scalar product $\langle Av_n, Av_n\rangle$. Therefore the global bounds of Eq. (4.3) for eigenvectors will be used in section 7 to obtain pointwise error bounds for arbitrary field quantities of elastic eigenstates.

5. LOWER BOUNDS FOR THE EIGENVALUES λ_n

The eigenvalues λ_n of the differential eigenvalue problem (2.5) are reciprocal to the eigenvalues μ_n of the integral eigenvalue problem

$$Gu - \mu u = 0 \qquad \mu = \frac{1}{\lambda} \qquad u \in L^2(\Omega) \quad , \tag{5.1}$$

where G is a completely continuous integral operator defined for all $u \in L^2(\Omega)$ by

$$Gu = \int_\Omega G(x,y)u(y)\,dV_y \; . \tag{5.2}$$

The kernel $G(x,y)$ is Green's function, for which

$$\iint_{\Omega\Omega} |G(x,y)|^2 dV_x dV_y < \infty \tag{5.3}$$

is assumed here.
The solution of the eigenvalue problem (5.1) is an infinite series of eigenvalues

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq \dots \tag{5.4}$$

with associated eigenvectors

$$u_1 \; ; \; u_2 \; ; \; \dots \; ; \; u_n \; ; \; \dots \quad , \tag{5.5}$$

corresponding to

$$Gu_n - \mu_n u_n = 0 \quad . \tag{5.6}$$

The operator G of Eq. (5.1) can be generalized by using iterated kernels $G^m(x,y)$

$$G^m u = \int_\Omega G^m(x,y) u(y) dV_y \tag{5.7}$$

leading to the eigenvalue equation

$$G^m u - \mu^m u = 0 \quad ; \tag{5.8}$$

The first invariant of the iterated kernel is given as

$$I_1(G^m) = \int_\Omega G^m(x,x) dV_x \quad , \tag{5.9}$$

which is for m = 1 the trace of Green's function $G(x,y)$.

As it is shown by Trefftz [1933] lower bounds for the eigenvalues λ_m and upper bounds for $\mu_n = \frac{1}{\lambda_n}$, respectively,

$$l_n \leq \lambda_n \quad ; \quad \mu_n \leq M_n \tag{5.10}$$

can be derived by using Rayleigh-Ritz approximations

$$\lambda_n \leq L_n \qquad m_n \leq \mu_n \quad , \tag{5.11}$$

if the first invariant of Green's function $G(x,y)$ or of an iterated kernel $G^m(x,y)$ is known.

Assuming the existence of $I_1(G)$ we have

$$I_1(G) = \int_\Omega G(x,x)\,dV_x = \sum_{\beta=1}^{\infty} \mu_\beta \tag{5.12}$$

yielding the inequality

$$I_1(G) - \sum_{\beta=1}^{\nu}{}^{(n)} \mu_\beta \geq \mu_n \quad , \tag{5.13}$$

where $\sum_\beta^{(n)}$ means that the term corresponding to the index n is suppressed from the summation. Replacing in the sum of Eq. (5.13) the eigenvalues μ_β by Rayleigh-Ritz bounds m_β upper eigenvalue bounds

$$M_n^{(\nu)} = I_1(G) - \sum_{\beta=1}^{\nu}{}^{(n)} m_\beta^{(\nu)} \geq \mu_n$$

with (5.14)

$$M_n^{(\nu)} \geq M_n^{(\nu+1)} \qquad \lim_{\nu\to\infty} M_n^{(\nu)} = \mu_n$$

are obtained.

This result was generalized by Fichera [1966], who derived eigenvalue bounds of the form (5.14) by introducing the orthogonal invariants of positive compact operators.

The crucial point in formula (5.14) is that an invariant of the completely continuous operator G or of an iterated operator G^m is not known in general, because the calculation of Green's function $G(x,y)$ is equivalent to the solution of the analogous boundary value problem. Therefore we have to contruct a decreasing sequence of upper invariant bounds

$$I_1(G_\rho) \geq I_1(G_{\rho+1}) \geq \ldots \geq I_1(G) \qquad \lim_{\rho\to\infty} I_1(G_\rho) = I_1(G) \quad , \tag{5.15}$$

which can be used to derive a converging sequence of upper eigenvalue bounds

$$M_n^{(\nu,\rho)} = I_1(G_\rho) - \sum_{\beta=1}^{\nu}{}^{(n)} m_\beta^{(\nu)} \geq \mu_n$$

$$M_n^{(\nu_1,\rho_1)} \geq M_n^{(\nu_2,\rho_2)} \qquad (\nu_1 \leq \nu_2 \; ; \; \rho_1 \leq \rho_2) \tag{5.16}$$

$$\lim_{\substack{\nu\to\infty \\ \rho\to\infty}} M_n^{(\nu,\rho)} = \mu_n \quad .$$

Estimates of the type (5.15) will be considered in the following section.

6. UPPER BOUNDS FOR THE INVARIANTS OF GREEN'S FUNCTION

Let us consider the elastic states of a linear elastic body as elements of a Hilbert space H. The scalar product of two elements $f,g \in H$ is defined by the interaction energy of the elastic states f and g:

$$\{f,g\} = \int_\Omega \varepsilon_{ij}(f) c_{ijkl} \varepsilon_{kl}(g) dV = \int_\Omega \sigma_{ij}(f) s_{ijkl} \sigma_{kl}(g) dV \quad , \tag{6.1}$$

where s_{ijkl} is the tensor of elastic coefficients, which is inverse to c_{ijkl}. From Eq. (6.1) follows the norm

$$\| f \| = \sqrt{\int_\Omega \varepsilon_{ij} c_{ijkl} \varepsilon_{kl} dV} \tag{6.2}$$

representing the double elastic energy of the state f.

The Hilbert space H can be decomposed into two orthogonal subspaces H' and H''

$$H = H' \oplus H'' \qquad H' \perp H'' \quad . \tag{6.3}$$

In (6.3) H' is the subspace of loadstress states $f' \in H'$, which can be derived from a displacement field satisfying homogeneous geometric boundary conditions on S_1, whereas H'' is the subspace of self-stress states $f'' \in H''$ satisfying the homogeneous equilibrium equation in Ω and homogeneous static boundary conditions on S_2. With these definitions

$$f = f' + f'' \quad ; \quad \{f',f''\} = 0 \tag{6.4}$$

is valid, from which (6.3) follows.

Let $f^{\sim}$ be a geometrically admissible approximation of f satisfying all geometrical conditions of f such that $(f^{\sim} - f)' \in H'$ and let $f^{\approx}$ be a statically admissible approximation of f such that $(f^{\approx} - f)'' \in H''$. Then the orthogonality condition $(6.3)_2$ leads to the following estimates:

$$\| f - f^{\sim} \| \leq \| f^{\sim} - f^{\approx} \|$$

$$\| f - f^{\approx} \| \leq \| f^{\sim} - f^{\approx} \|$$

$$\| f - \tfrac{1}{2} (f^{\sim} + f^{\approx}) \| = \tfrac{1}{2} \| f^{\sim} - f^{\approx} \| \quad . \qquad (6.5)$$

To derive upper bounds for the invariants of a Green's function $G(x,y)$ we introduce elastic Green's states $\overset{0}{f}(y) \in H$, which are characterized by a unit geometrical or statical singularity at the point $y \in \bar{\Omega}$ satisfying for $x \neq y$ homogeneous equilibrium and compatibility equations and on the boundary S_1 homogeneous geometric and on the boundary S_2 homogeneous static conditions. At this moment it is assumed that $\| \overset{0}{f}(y) \| < \infty$.

The Green's function $G(x,y)$ can be interpreted physically as the displacement field at a point $x \in \bar{\Omega}$ of an elastic Green's state $\overset{0}{f}(y)$ produced by a unit force at $y \in \Omega$. Therefore $\overset{0}{f}(y)$ is a loadstress state $\overset{0}{f}{}'(y) \in H'$.
Using two Green's states $\overset{0}{f}(x)$ and $\overset{0}{f}(y)$ with unit forces at x and y, respectively, we have:

$$G(x,y) = \{\overset{0}{f}(x) \, , \, \overset{0}{f}(y)\} \qquad G(x,x) = \| \overset{0}{f}(x) \|^2 \quad . \qquad (6.6)$$

In general the stress, strain or displacement field of a Green's state $\overset{0}{f}(x)$ is unknown. So we choose a statically admissible approximation $\overset{0}{f}{}^{\approx}(x)$

$$\overset{0}{f}{}^{\approx}(x) = \overset{0}{f}{}'(x) + (\overset{0}{f}{}^{\approx}(x) - \overset{0}{f}{}'(x))'' \qquad (6.7)$$

yielding upper bounds for $\| \overset{0}{f}(x) \|$.

$$G^{\approx}(x,x) = \| \overset{0}{f}{}^{\approx}(x) \|^2 \geq \| \overset{0}{f}(x) \|^2 = G(x,x) \quad . \qquad (6.8)$$

In order to obtain a decreasing sequence of upper bounds $\| \overset{0}{f}{}^{\approx}(x) \|$ we use a first approximation $\overset{0}{f}{}^{\approx}_{o}(x)$ and superpose a complete system of

ρ linearly independent selfstress states by using stress functions with unknown coefficients, which can be computed by a minimizing process. This leads to

$$G_\rho(x,x) \geq G_{\rho+1}(x,x) \geq \ldots \geq G(x,x) \qquad \lim_{\rho\to\infty} G_\rho(x,x) = G(x,x). \tag{6.9}$$

With relation (6.9) and Eq. (5.12) a decreasing sequence of upper invariant bounds (5.15) can be calculated.

7. POINTWISE ERROR BOUNDS FOR ARBITRARY FIELD QUANTITIES OF EIGENSTATES

In section 5 the eigenvibration problem of linear elastic bodies had been characterized by Green's integral operator G defined by Eq. (5.2) on the space of square-integrable displacement functions $u \in L^2(\Omega)$. Analogously a completely continuous operator $\mathcal{B}$ on the Hilbert-space H' of loadstress states f' is constructed by the statement that for all $f \in H'$ the elastic states $\mathcal{B}f$ are defined by body forces b proportional to the displacement u of f and homogeneous geometric and static boundary conditions:

$$\begin{aligned} &f &&: u(x) && \forall x \in \bar{\Omega} \\ &\hat{f} = \mathcal{B}f &&: \hat{b}(x) = Cu(x) && \forall x \in \Omega\ , \end{aligned} \tag{7.1}$$

where C is the mass density devided by the square of a time coefficient. It can be shown easily that the operator $\mathcal{B}$ is symmetric:

$$\begin{aligned} \{\mathcal{B}f_1, f_2\} &= \{\hat{f}_1, f_2\} \\ &= \int_\Omega \varepsilon_{ij}(\hat{u}_1) c_{ijkl} \varepsilon_{kl}(u_2)\, dV \\ &= \int_\Omega \hat{b}_i^{(1)} u_i^{(2)}\, dV = \int_\Omega C u_i^{(1)} u_i^{(2)}\, dV \\ &= \{f_1, \mathcal{B}f_2\}\ . \end{aligned} \tag{7.2}$$

With the symmetric operator $\mathcal{B}$ the eigenvibration problem is defined by

$$\mathcal{B}f - \mu f = 0 \qquad f \in H'\ , \tag{7.3}$$

which corresponds to the eigenvalue problem (5.1). The solution of (7.3) consists of an infinite series of eigenvalues μ_n

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq \ldots \rightarrow 0 \tag{7.4}$$

with associated eigenstates $\overset{<n>}{f}$ satisfying

$$\mathcal{B}\overset{<n>}{f} - \mu_n \overset{<n>}{f} = 0 \qquad \{\overset{<n>}{f}, \overset{<m>}{f}\} = \delta_{nm} \,. \tag{7.5}$$

The eigenvalues μ_n are the stationary values of the Rayleigh-quotient $\mu\{f\}$ for all $f' \in \mathcal{H}'$:

$$\mu\{f\} = \frac{\{\mathcal{B}f, f\}}{\{f,f\}} = \frac{\{\hat{f}, f\}}{\{f,f\}}$$
$$= \frac{\int\limits_\Omega \varepsilon_{ij}(\hat{u}) c_{ijkl} \varepsilon_{kl}(u) dV}{\int\limits_\Omega \varepsilon_{ij}(u) c_{ijkl} \varepsilon_{kl}(u) dV} = \frac{\int\limits_\Omega C u_i u_i \, dV}{\int\limits_\Omega \varepsilon_{ij}(u) c_{ijkl} \varepsilon_{kl}(u) dV} \,, \tag{7.6}$$

where $\mu\{f\}$ is reciprocal to the Rayleigh-quotient λ of the differential operator $\mathcal{A}$ according to Eq. (2.14).

Repeated application of the operator $\mathcal{B}$ yields the eigenvalue equation $(7.5)_1$ in the form

$$\mathcal{B}^m \overset{<n>}{f} - \mu_n^m \overset{<n>}{f} = 0 \,, \tag{7.7}$$

where $\mathcal{B}^m$ is the m-th iterated operator.

The eigenstates $\overset{<n>}{f}$ can be represented by their displacement field u_n or by the associated stress or strain fields. In order to calculate pointwise bounds for an arbitrary field quantity of $\overset{<n>}{f}$ we use Green's states $\overset{0}{f}(x)$ with an appropriate geometric or static singularity at a point $x \in \Omega$. Besides the singularity the Green's states are characterized by vanishing body forces and selfstress sources, whereas the geometric and static boundary conditions are homogeneous. The singularity of $\overset{0}{f}(x)$ has to be chosen such that the scalar product $\{\overset{<n>}{f}, \overset{0}{f}(x)\}$ yields the field quantity $\overset{\langle n\rangle}{F}(x)$ for the n-the eigenstate $\overset{\langle n\rangle}{f}$ at a point $x \in \Omega$:

$$\overset{\langle n\rangle}{F}(x) = \{\overset{\langle n\rangle}{f}, \overset{0}{f}(x)\} \quad . \tag{7.8}$$

In general the eigenstates $\overset{\langle n\rangle}{f}$ and the Green's states $\overset{0}{f}(x)$ are unknown. Therefore we derive a formula to calculate pointwise bounds for the field quantity sought.

If the Green's state $\overset{0}{f}(x)$ has no finite energy, we use the m-th iterated operator $\mathcal{B}^m$ with positive integers m such that $\| \mathcal{B}^m\overset{0}{f}(x) \| < \infty$. Corresponding to (7.8) we obtain

$$\overset{\langle n\rangle}{F}(x) = \frac{1}{\mu_n^m}\{\overset{\langle n\rangle}{f}, \mathcal{B}^m\overset{0}{f}(x)\} \qquad \text{m positive integer.} \tag{7.9}$$

To estimate the scalar product of the right side of (7.8) and (7.9), respectively, the eigenstates $\overset{\langle n\rangle}{f}$ can be approximated by Rayleigh-Ritz eigenstates $\overset{n}{f} \in \mathcal{H}'$, which are assumed to be normalized with $\|\overset{n}{f}\| = 1$. Then Schwarz' inequality yields:

$$|\{\overset{\langle n\rangle}{f}, \mathcal{B}^m\overset{0}{f}(x)\} - \{\overset{n}{f}, \mathcal{B}^m\overset{0}{f}(x)\}| \le \|\overset{\langle n\rangle}{f} - \overset{n}{f}\| \, \|\mathcal{B}^m\overset{0}{f}(x)\| \,. \tag{7.10}$$

With (7.10) the field quantity $\overset{\langle n\rangle}{F}(x)$ according to (7.9) can be estimated and we obtain the following result:

$$\left|\overset{\langle n\rangle}{F}(x) - \frac{1}{\mu_n^m}\{\overset{n}{f}, \mathcal{B}^m\overset{0}{f}(x)\}\right| \le \frac{1}{\mu_n^m} \|\overset{\langle n\rangle}{f} - \overset{n}{f}\| \, \|\mathcal{B}^m\overset{0}{f}(x)\| \quad . \tag{7.11}$$

The application of formula (7.11) for pointwise bounding of $\overset{\langle n\rangle}{F}(x)$ consists now of four parts:

the computation of upper and lower bounds for the eigenvalues μ_n, which had been considered in section 3 and 5; the derivation of an upper bound for $\|\overset{\langle n\rangle}{f} - \overset{n}{f}\|$ expressing the difference in norm of the eigenstate $\overset{\langle n\rangle}{f}$ and its Rayleigh-Ritz approximation, given in section 4; an estimation of the norm $\| \mathcal{B}^m\overset{0}{f}(x) \|$ and the calculation of bounds for the scalar product $\{\overset{n}{f}, \mathcal{B}^m\overset{0}{f}(x)\}$, which will be considered in this section, separately for the cases $m = 0 : \| \overset{0}{f}(x) \| < \infty$, $m = 1 : \| \mathcal{B}\overset{0}{f}(x) \| < \infty$, m arbitrary positive integer with: $\| \mathcal{B}^m\overset{0}{f}(x) \| < \infty$.

a) $m = 0 : \|\overset{0}{f}(x)\| < \infty$

We approximate the unknown Green's state $\overset{0}{f}(x)$ by $\frac{1}{2}(\overset{0\sim}{f}(x) + \overset{0\approx}{f}(x))$. With $\overset{0\sim}{f}(x)$ we denote a geometrically admissible approximation of $\overset{0}{f}(x)$ satisfying all geometrical conditions of $\overset{0}{f}(x)$, the compatibility condition and the geometric boundary conditions. Analogously let $\overset{0\approx}{f}(x)$ be a statically admissible approximation of $\overset{0}{f}(x)$ satisfying all statical conditions of $\overset{0}{f}(x)$, the equilibrium equation and the static boundary conditions. Then the orthogonality condition (6.5) leads to the estimation:

$$\|\overset{0}{f}(x)\| \le \frac{1}{2}\|\overset{0\sim}{f}(x) + \overset{0\approx}{f}(x)\| + \frac{1}{2}\|\overset{0\approx}{f}(x) - \overset{0\sim}{f}(x)\| . \tag{7.12}$$

If $\overset{n}{f}$ is a Rayleigh-Ritz approximation satisfying the geometric boundary conditions of $\overset{\langle n\rangle}{f}$, the scalar product

$$\{\overset{n}{f}, \overset{0}{f}(x)\} = \overset{n}{F}(x) \tag{7.13}$$

yields the known field quantity $\overset{n}{F}(x)$ of $\overset{n}{f}$, which can be used in (7.11) as approximation for the unknown exact field quantity $\overset{\langle n\rangle}{F}(x)$ of the n-th eigenstate $\overset{\langle n\rangle}{f}$.

b) $m = 1 : \|B\overset{0}{f}(x)\| < \infty$

For $\overset{0}{f}(x) \in H'$ we can put:

$$\overset{0}{f}(x) = \overset{0\sim}{f}(x) + g_o(x) \qquad \overset{0\sim}{f}(x), g_o(x) \in H' . \tag{7.14}$$

By application of B we obtain

$$B\overset{0}{f}(x) = B\overset{0\sim}{f}(x) + Bg_o(x) . \tag{7.15}$$

The last term on the right side of Eq. (7.15) can be estimated as follows:

$$\|Bg_o(x)\| \le \|B\|\,\|g_o(x)\| \le \mu_1\|\overset{0\approx}{f}(x) - \overset{0\sim}{f}(x)\| \le M_1\|\overset{0\approx}{f}(x) - \overset{0\sim}{f}(x)\| , \tag{7.16}$$

where μ_1 is the largest eigenvalue of the operator B and $M_1 \ge \mu_1$ an upper bound for μ_1.

The elastic state $\mathcal{B}\overset{0}{f}{}^{\sim}(x)$ is defined by a body force density proportional to the known displacement field of the geometrically admissible approximation $\overset{0}{f}{}^{\sim}(x)$. Hence we use the following approximation:

$$\mathcal{B}\overset{0}{f}{}^{\sim}(x) = (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} + g_1(x) \quad , \qquad (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim}, g_1(x) \in H' , \tag{7.17}$$

where an upper bound for $g_1(x)$ can be obtained:

$$\| g_1(x) \| \leq \| (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\approx} - (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} \| \quad . \tag{7.18}$$

With the inequalities (7.16) and (7.18) we derive an upper bound for the norm of $\mathcal{B}\overset{0}{f}(x)$:

$$\begin{aligned} \| \mathcal{B}\overset{0}{f}(x) \| \leq{} & \| (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} \| + \| (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\approx} - (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} \| \\ & + M_1 \| \overset{0}{f}{}^{\approx}(x) - \overset{0}{f}{}^{\sim}(x) \| \quad . \end{aligned} \tag{7.19}$$

Furthermore we have to estimate the scalar product $\{\overset{n}{f}, \mathcal{B}\overset{0}{f}(x)\}$ in formula (7.11). We approximate the unknown state $\mathcal{B}\overset{0}{f}(x)$ by $(\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim}$ and obtain with (7.15) and (7.17):

$$\mathcal{B}\overset{0}{f}(x) = (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} + g_1(x) + \mathcal{B}g_o(x) \quad , \tag{7.20}$$

which enables the calculation of global bounds yielding with (7.16) and (7.18):

$$\begin{aligned} |\{\overset{n}{f}, \mathcal{B}\overset{0}{f}(x)\} - \{\overset{n}{f}, (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim}\}| &= |\{\overset{n}{f}, \mathcal{B}\overset{0}{f}(x) - (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim}\}| \\ &\leq \| \overset{n}{f} \| \, \| \mathcal{B}\overset{0}{f}(x) - (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} \| \\ &\leq \| (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\approx} - (\mathcal{B}\overset{0}{f}{}^{\sim}(x))^{\sim} \| + M_1 \| \overset{0}{f}{}^{\approx}(x) - \overset{0}{f}{}^{\sim}(x) \| , \end{aligned}$$

where normalized Rayleigh-Ritz approximations $\overset{n}{f}$ with $\| \overset{n}{f} \| = 1$ are used.

c) <u>m positive interger such that $\| B^m \overset{0}{f}(x) \| < \infty$</u>

Now we consider the general case with m a positive integer such that the norm of $B^m \overset{0}{f}(x)$ is finite. According to Stumpf [1970] and [1972c] we proceed as follows:

$$\underline{m = 0} : \overset{0}{f}(x) = \overset{0}{f}{}^{\sim}(x) + g_o(x)$$

$$= \varphi_o(x) + g_o(x) \tag{7.22}$$

$$\| \overset{0}{f}(x) \| \leq \| \varphi_o(x) \| + \| \overset{0}{f}{}^{\approx}(x) - \overset{0}{f}{}^{\sim}(x) \| \tag{7.23}$$

$$\underline{m = 1} : B\overset{0}{f}(x) = B\varphi_o(x) + Bg_o(x)$$

$$= (B\varphi_o(x))^{\sim} + g_1(x) + Bg_o(x)$$

$$= \varphi_1(x) + g_1(x) + g_o(x) \tag{7.24}$$

$$\| B\overset{0}{f}(x) \| \leq \| \varphi_1(x) \| + \| (B\varphi_o(x))^{\approx} - (B\varphi_o(x))^{\sim} \|$$

$$+ M_1 \| \overset{0}{f}{}^{\approx}(x) - \overset{0}{f}{}^{\sim}(x) \| \tag{7.25}$$

$$\underline{m = 2} : B^2\overset{0}{f}(x) = B\varphi_1(x) + Bg_1(x) + B^2 g_o(x)$$

$$= (B\varphi_1(x))^{\sim} + g_2(x) + Bg_1(x) + B^2 g_o(x)$$

$$= \varphi_2(x) + g_2(x) + Bg_1(x) + B^2 g_o(x) \tag{7.26}$$

$$\| B^2 f(x) \| \leq \| \varphi_2(x) \| + \| (B\varphi_1(x))^{\approx} - (B\varphi_1(x))^{\sim} \|$$

$$+ M_1 \| (B\varphi_o(x))^{\approx} - (B\varphi_o(x))^{\sim} \|$$

$$+ M_1^2 \| \overset{0}{f}{}^{\approx}(x) - \overset{0}{f}{}^{\sim}(x) \| \tag{7.27}$$

$$\underline{m > 2} : B^m\overset{0}{f}(x) = B\varphi_{m-1}(x) + Bg_{m-1}(x) + B^2 g_{m-2}(x) + \ldots + B^m g_o(x)$$

$$= (B\varphi_{m-1}(x))^{\sim} + g_m(x) + Bg_{m-1}(x) + \ldots + B^m g_o(x)$$

$$= \varphi_m(x) + g_m(x) + Bg_{m-1}(x) + \ldots + B^m g_o(x) \tag{7.28}$$

$$\| \overset{0}{\mathcal{B}^m f}(x) \| \leq \| \varphi_m(x) \| + \| (\mathcal{B}\varphi_{m-1}(x))^{\approx} - (\mathcal{B}\varphi_{m-1}(x))^{\sim} \|$$
$$+ M_1 \| (\mathcal{B}\varphi_{m-2}(x))^{\approx} - (\mathcal{B}\varphi_{m-2}(x))^{\sim} \|$$
$$+ \dots + M_1^m \| \overset{0\approx}{f}(x) - \overset{0\sim}{f}(x) \| \quad . \tag{7.29}$$

Analogously we estimate the scalar product $\{\overset{n}{f}, \mathcal{B}^m \overset{0}{f}(x)\}$ in formula (7.11) for arbitrary positive integer m:

$$|\{\overset{n}{f}, \mathcal{B}^m \overset{0}{f}\, x\} - \{\overset{n}{f}, \varphi_m(x)\}| = |\{\overset{n}{f}, \mathcal{B}^m \overset{0}{f}(x) - \varphi_m(x)\}|$$
$$\leq \| \overset{n}{f} \| \; \| \mathcal{B}^m \overset{0}{f}(x) - \varphi_m(x) \|$$
$$\leq \| (\mathcal{B}\varphi_{m-1}(x))^{\approx} - (\mathcal{B}\varphi_{m-1}(x))^{\sim} \| +$$
$$+ M_1 \| (\mathcal{B}\varphi_{m-2}(x))^{\approx} - (\mathcal{B}\varphi_{m-2}(x))^{\sim} \| +$$
$$+ \dots + M_1^m \| f_o^{\approx}(x) - f_o^{\sim}(x) \| \, . \tag{7.30}$$

With the estimates (7.29) and (7.30), with given upper bounds $\| \overset{<n>}{f} - \overset{n}{f} \|$ and upper and lower bounds for the eigenvalues μ_n, two-sided pointwise error bounds for arbitrary field quantities of an arbitrary eigenstate $\overset{<n>}{f}$ can be calculated by application of formula (7.11). In order to improve the numerical results we choose zero-approximations $(\mathcal{B}\varphi_k(x))_o^{\sim}$ and $(\mathcal{B}\varphi_k(x))_o^{\approx}$ and superpose a complete system of linearly independent loadstress states and selfstress states with unknown coefficients, which can be determined by minimizing processes.

8. APPLICATION TO CLAMPED RECTANGULAR ORTHOTROPIC PLATES

To describe the eigenvibrations of a clamped rectangular orthotropic plate, we use a Cartesian coordinate system $(\bar{x}, \bar{y}, \bar{z})$, where the $\bar{x},\bar{y}$-plane coincide with the undeformed plate middle surface. The equation of motion can be given in the form

$$\bar{D}_x \frac{\partial^4 w}{\partial \bar{x}^4} + 2\bar{H} \frac{\partial^4 w}{\partial \bar{x}^2 \partial \bar{y}^2} + \bar{D}_y \frac{\partial^4 w}{\partial \bar{y}^4} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{8.1}$$

with the boundary conditions:

$$w = \frac{\partial w}{\partial \bar{x}} = 0 \quad \text{for } \bar{x} = 0,a$$
$$w = \frac{\partial w}{\partial \bar{y}} = 0 \quad \text{for } \bar{y} = 0,b \; , \tag{8.2}$$

where w is the diplacement of the middle surface, ρ the mass density and h the plate thickness.

The rigidities are defined by

$$\bar{D}_x = \frac{[EJ]_x}{1-\nu_x\nu_y} \; ; \; \bar{D}_y = \frac{[EJ]_y}{1-\nu_x\nu_y} \; ; \; \bar{D}_{xy} = \frac{1}{2}\,(1-\sqrt{\nu_x\nu_y})\,\sqrt{\bar{D}_x\bar{D}_y} \; . \tag{8.3}$$

In the formulas (8.3) E is Young's modulus, J the moment of inertia and ν Poisson's ratio. Also the definitions are introduced:

$$\bar{D}_1 = \frac{1}{2}\,(\bar{D}_x\nu_y + \bar{D}_y\nu_x) \; ; \; \bar{H} = \bar{D}_1 + 2\bar{D}_{xy} \; ; \; \mu = \frac{\bar{H}}{\sqrt{\bar{D}_x\bar{D}_y}} \; . \tag{8.4}$$

For the special case of isotropic plates we have

$$\nu_x = \nu_y = \nu \; ; \quad \bar{D}_x = \bar{D}_y = N \doteq \frac{Eh^3}{12(1-\nu^2)} \; ; \; \bar{D}_1 = \nu N \; ; \; \bar{D}_{xy} = \frac{1-\nu}{2}\,N \; . \tag{8.5}$$

In the following formulas normed coefficients $D = \bar{D}/N$, $H = \bar{H}/N$ and normed coordinates $x = \bar{x}/a$, $y = \bar{y}/b$ will be used.

The solution of (8.1) is assumed in the following form

$$w(x,y,t) = u(x,y)\,\cos\,(\omega t + \alpha) \tag{8.6}$$

yielding the eigenvalue problem

$$D_x\,\frac{\partial^4 u}{\partial x^4} + 2H\left(\frac{a}{b}\right)^2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + D_y\left(\frac{a}{b}\right)^4 \frac{\partial^4 u}{\partial y^4} - \lambda u = 0$$
$$u = \frac{\partial u}{\partial x} = 0 \quad \text{for} \quad x = 0,1 \quad ; \quad u = \frac{\partial u}{\partial y} = 0 \quad \text{for} \quad y = 0,1 \tag{8.7}$$

with the eigenvalues

$$\lambda = \rho h a^4 \omega^2 / N \; . \tag{8.8}$$

$$Au = D_x \frac{\partial^4 u}{\partial x^4} + 2H\left(\frac{a}{b}\right)^2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + D_y\left(\frac{a}{b}\right)^4 \frac{\partial^4 u}{\partial y^4} \tag{8.9}$$

is symmetric and positive definite for all u satisfying $(8.7)_2$. Therefore the methods described in this paper can be applied to obtain approximate solutions.

The scalar products $(2.7)_1$ and (2.10) yield with (8.9)

$$\langle u,v\rangle = \int_0^1\int_0^1 uv\,dxdy \tag{8.10}$$

$$\langle Au,v\rangle = \{u,v\} =$$

$$= \int_0^1\int_0^1 [D_x u_{,xx}v_{,xx} + D_y\left(\frac{a}{b}\right)^4 u_{,yy}v_{,yy} + D_1\left(\frac{a}{b}\right)^2(u_{,xx}v_{,yy} + u_{,yy}v_{,xx})$$

$$+ 4D_{xy}\left(\frac{a}{b}\right)^2 u_{,xy}v_{,xy}]\,dxdy \tag{8.11}$$

where $\{u,v\}$ is the interaction energy of two elastic states represented by their displacement fields u and v.

With the constitutive relations

$$M_{xx} = \bar{D}_1 \frac{\partial^2 u}{\partial \bar{x}^2} + \bar{D}_y \frac{\partial^2 u}{\partial \bar{y}^2}$$

$$M_{yy} = \bar{D}_x \frac{\partial^2 u}{\partial \bar{x}^2} + \bar{D}_1 \frac{\partial^2 u}{\partial \bar{y}^2} \tag{8.12}$$

$$M_{xy} = -2\bar{D}_{xy} \frac{\partial^2 u}{\partial \bar{x}\partial \bar{y}}$$

the interaction energy $\{\overset{1}{f},\overset{2}{f}\}$ can also be expressed by

$$\{\overset{1}{f},\overset{2}{f}\} = \frac{a^4/N^2}{D_xD_y-D_1^2}\int_0^1\int_0^1 [D_x\overset{1}{M}_{xx}\overset{2}{M}_{xx} + D_y\overset{1}{M}_{yy}\overset{2}{M}_{yy} - D_1(\overset{1}{M}_{xx}\overset{2}{M}_{yy} + \overset{1}{M}_{yy}\overset{2}{M}_{xx})$$

$$+ \frac{D_xD_y-D_1^2}{D_{xy}}\overset{1}{M}_{xy}\overset{2}{M}_{xy}]\,dxdy \quad . \tag{8.13}$$

To calculate Raleigh-Ritz upper eigenvalue bounds we approximate the displacement field by a function

$$u = \sum_{\alpha=1}^{\nu} a_\alpha \varphi_\alpha \qquad (8.14)$$

with coordinate functions φ_α of the form

$$x^2(1-x)^2 y^2 (1-y)^2 x^i x^j \quad i,j \text{ integers.} \qquad (8.15)$$

The minimizing procedure described in section 3 yields upper eigenvalue bounds $L_n^{(\nu)}$ (table 1) and associated Rayleigh-Ritz approximations of the eigenfunctions.

To derive lower eigenvalue bounds and pointwise error bounds for the eigenvectors we use the Green's function $\overset{0}{u}(x_o,y_o)$ of a singular force acting at a point (x_o,y_o) of the plate. Usually the Green's function $\overset{0}{u}$ is splitted into a fundamental part $\overset{\infty}{u}$ and a regular part $\overset{\vee}{u}$:

$$\overset{0}{u}(x_o,y_o) = \overset{\infty}{u}(x_o,y_o) + \overset{\vee}{u}(x_o,y_o) \quad , \qquad (8.16)$$

where the fundamental part for $\mu = H/\sqrt{D_x D_y}$ is given by Stein [1959]:

$$\chi = (\bar{x} - \bar{x}_o)/\sqrt[4]{D_x} \qquad\qquad y = (\bar{y} - \bar{y}_o)/\sqrt[4]{D_y}$$

$\mu = 0$

$$\overset{\infty}{u}(\chi,y) = \frac{1}{8\pi\sqrt[4]{D_x D_y}}\left[\frac{\chi^2+y^2}{\sqrt{2}} \ln\sqrt{\chi^4+y^4} + 2\chi y \ln\sqrt{\frac{\chi^2+\sqrt{2}\chi y+y^2}{\chi^2-\sqrt{2}\chi y+y^2}}\right.$$

$$\left.+ \frac{1}{\sqrt{2}}\left\{\chi^2 \arctan\frac{\chi^2}{y^2} + y^2 \arctan\frac{y^2}{\chi^2}\right\}\right]$$

$0 < \mu < 1$

$$\overset{\infty}{u}(\chi,y) = \frac{1}{8\pi\sqrt[4]{D_x D_y}}\left[\frac{\chi^2+y^2}{\sqrt{2(1+\mu)}} \ln\sqrt{\chi^4+2\mu\chi^2 y^2+y^4}\right.$$

$$+ 2\frac{\chi y}{\sqrt{1-\mu^2}} \ln\sqrt{\frac{\chi^2+\sqrt{2(1-\mu)}\chi y+y^2}{\chi^2-\sqrt{2(1-\mu)}\chi y+y^2}}$$

$$\left.+ \frac{1}{\sqrt{2(1-\mu)}}\left\{\chi^2 \arctan\frac{\sqrt{1-\mu^2}\,\chi^2}{y^2+\mu\chi^2} + y^2 \arctan\frac{\sqrt{1-\mu^2}\,\chi^2}{\chi^2+\mu y^2}\right\}\right]$$

$\underline{\mu = 1}$

$$\overset{\infty}{u}(x,y) = \frac{1}{8\pi\sqrt[4]{D_x D_y}} \, (x^2 + y^2)\,\{1 + \ln\sqrt{x^2+y^2}\}$$

$\underline{1 < \mu < \infty}$

$$\overset{\infty}{u}(x,y) = \frac{1}{8\pi\sqrt[4]{D_x D_y}} \Big[\frac{x^2+y^2}{\sqrt{2(1+\mu)}} \ln\sqrt{x^4 + 2\mu x^2 y^2 + y^4}$$

$$+ 2\,\frac{xy}{\sqrt{\mu^2-1}} \arctan\frac{\sqrt{2(\mu-1)}\,xy}{x^2+y^2} \; +$$

$$+ \; \frac{1}{\sqrt{2(\mu-1)}}\{x^2 \ln\sqrt{\frac{(\mu+\sqrt{\mu^2-1})\,x^2+y^2}{(\mu-\sqrt{\mu^2-1})\,x^2+y^2}}$$

$$+ y^2 \ln\sqrt{\frac{x^2+(\mu+\sqrt{\mu^2-1})\,y^2}{x^2+(\mu-\sqrt{\mu^2-1})\,y^2}}\;\}\Big] \tag{8.17}$$

With (8.16) we approximate the unknown Green's state

$$\overset{0}{f}(x_o,y_o) = \overset{\infty}{f}(x_o,y_o) + \overset{\vee}{f}(x_o,y_o) \tag{8.18}$$

by a geometrically admissible approximation $\overset{0}{f}{}^{\sim}(x_o,y_o)$

$$\overset{0}{f}{}^{\sim}(x_o,y_o) \; : \; \overset{0}{u}{}^{\sim}(x_o,y_o) = \overset{\infty}{u}(x_o,y_o) + \overset{\vee}{u}{}^{\sim}(x_o,y_o) \; , \tag{8.19}$$

where $\overset{\vee}{u}{}^{\sim}(x_o,y_o)$ has to satisfy the geometrical boundary conditions of the regular part $\overset{\vee}{u}(x_o,y_o)$. For the numerical computations of $\overset{\vee}{u}{}^{\sim}(x_o,\, y_o)$ a complete system of coordinate functions is used.

Representing the Green's state $\overset{0}{f}(x_o,y_o)$ by the stress couple tensor

$$\overset{0}{M}(x_o,y_o) = \begin{pmatrix} \overset{0}{M}_{xx}(x_o,y_o) & \overset{0}{M}_{xy}(x_o,y_o) \\ \overset{0}{M}_{yx}(x_o,y_o) & \overset{0}{M}_{yy}(x_o,y_o) \end{pmatrix} \tag{8.20}$$

we have equivalent to (8.16):

$$\overset{0}{M}(x_o,y_o) = \overset{\infty}{M}(x_o,y_o) + \overset{\vee}{M}(x_o,y_o) , \tag{8.21}$$

where the fundamental part $\overset{\infty}{M}$ follows from (8.17) by (8.12). With (8.21) the unknown Green's state $\overset{0}{f}(x_o,y_o)$ can be approximated also by a statically admissible approximation

$$\overset{0}{f}{}^{\approx}(x_o,y_o) \quad : \quad \overset{0}{M}{}^{\approx}(x_o,y_o) = \overset{\infty}{M}(x_o,y_o) + \overset{\vee}{M}{}^{\approx}(x_o,y_o) \quad , \tag{8.22}$$

where $\overset{\infty}{M}(x_o,y_o)$ follows from (8.17) by (8.12), and $\overset{\vee}{M}{}^{\approx}$ has to satisfy the homogeneous equilibrium equations and the statical conditions of $\overset{\vee}{M}$. It is convenient to derive $\overset{\vee}{M}{}^{\approx}$ from stress functions Φ,Ψ:

$$\overset{\vee}{M}{}^{\approx}_{xx} = \frac{\partial\Phi}{\partial x} \quad ; \quad \overset{\vee}{M}{}^{\approx}_{yy} = \frac{\partial\Psi}{\partial y} \quad ; \quad \overset{\vee}{M}{}^{\approx}_{xy} = \overset{\vee}{M}{}^{\approx}_{yx} = \frac{1}{2}\left(\frac{\partial\Psi}{\partial x} + \frac{\partial\Phi}{\partial y}\right) \quad . \tag{8.23}$$

Representing the stress functions Φ,Ψ by a complete system of coordinate functions

$$\begin{aligned} \Phi &= \sum_{i,j} a_{ij}(x-x_o)^i(y-y_o)^j \\ \Psi &= \sum_{i,j} b_{ij}(x-x_o)^i(y-y_o)^j \end{aligned} \tag{8.24}$$

the coefficients a_{ij}, b_{ij} can be computed by minimizing

$$\begin{aligned} \| \overset{0}{f}{}^{\approx}(x_o,y_o) \|^2 = \frac{a^2/N}{D_xD_y - D_1^2} \int_0^1\!\!\int_0^1 [D_x \overset{0}{M}{}^{\approx 2}_{xx} + D_y \overset{0}{M}{}^{\approx 2}_{yy} - 2D_1 \overset{0}{M}{}^{\approx}_{xx} \overset{0}{M}{}^{\approx}_{yy} + \\ + \frac{D_xD_y - D_1^2}{D_{xy}} \overset{0}{M}{}^{\approx 2}_{xy}]dxdy \geq \| \overset{0}{f}(x_o,y_o) \|^2 , \end{aligned} \tag{8.25}$$

which is an estimate of the form (6.8) and which enables the calculation of lower eigenvalue bounds.

According to the formulas given in section 7 the approximations (8.19) and (8.22) will also be used to derive pointwise error bounds for the first eigenfunction $\overset{<1>}{u}$ and for the stress couples of the first eigenstate $\overset{<1>}{f}$.

9. NUMERICAL RESULTS

The computations are carried out for a/b = 1 and

$$D_x = 1{,}52439 \; ; \; D_y = 0{,}152439 \; ; \; D_{xy} = 0{,}218161 \; ; \; D_1 = 0{,}0457317 \; ; \; \mu = 1 \; . \tag{9.1}$$

Using 144 coordinate functions for the Rayleigh-Ritz approximation and 98 coordinate functions for the estimation of the first invariant (5.12) the following results are obtained:

$$\sum_{\beta=1}^{144} m_\beta = \sum_{\beta=1}^{144} \frac{1}{L_\beta} = 0{,}0029096 \; ; \; I_1(G_{98}) = 0{,}0029125 \; . \tag{9.2}$$

Upper Rayleigh-Ritz bounds and lower bounds according to the methods described in section 5 are given in table 1 for the first 10 eigenvalues:

n	l_n	L_n	n	l_n	L_n
1	977,07	979,84	6	8710,80	8936,55
2	1865,05	1875,19	7	11941,72	12370,11
3	4098,36	4147,66	8	17024,17	17908,31
4	6276,87	6393,25	9	18497,97	19546,52
5	8163,27	8361,20	10	22089,68	23601,60

Table 1

With these eigenvalue bounds formula (4.3) yields a global bound for the first eigenstate

$$\| \overset{<1>}{f} - \overset{1}{f} \| \leq 0{,}015483 \quad . \tag{9.3}$$

Let $\overset{<1>}{u}$ be the first eigenfunction and $\overset{1}{u}$ a corresponding Rayleigh-Ritz approximation. With $\Delta\overset{1}{u}$ we denote an upper bound for the difference

$$| \overset{<1>}{u}(x_o,y_o) - \overset{1}{u}(x_o,y_o) | \leq \Delta\overset{1}{u}(x_o,y_o) \quad , \tag{9.4}$$

the following pointwise error bounds for the first eigenfunction $\overset{<1>}{u}(x_o,y_o)$ have been obtained (for reason of symmetry the results are given for one quarter of the plate):

x_o	y_o	$\overset{1}{u}$	$\Vert\overset{0}{f}{}^{\approx}(x_o,y_o)\Vert$	$\overset{1}{\Delta u}$	$\overset{1}{\Delta u}/\overset{1}{u}$
0,5	0,5	0,0779	0,09143	0,001416	1,82 %
0,5	0,6	0,0723	0,09221	0,001428	1,97 %
0,5	0,7	0,0566	0,08687	0,001345	2,38 %
0,5	0,8	0,0343	0,07101	0,001099	3,21 %
0,5	0,9	0,0107	0,04327	0,000670	6,26 %
0,6	0,5	0,0716	0,08914	0,001380	1,93 %
0,7	0,5	0,0543	0,07528	0,001166	2,15
0,8	0,5	0,0310	0,06078	0,000941	3,04 %
0,9	0,5	0,0089	0,03923	0,000607	6,82 %
0,6	0,6	0,0664	0,08820	0,001366	2,06 %
0,7	0,7	0,0393	0,07202	0,001115	2,84 %
0,8	0,8	0,0134	0,06545	0,001013	7,56 %
0,9	0,9	0,00018	0,003028	0,000047	26,05 %

Table 2

Furthermore pointwise error bounds for the stress couple $\overset{<1>}{M}_{xx}$ of the first eigenstate $\overset{<1>}{f}$

$$\left| \overset{<1>}{M}_{xx}(x_o,y_o) - \overset{1}{M}_{xx}(x_o,y_o) \right| \leq \Delta\overset{1}{M}_{xx}(x_o,y_o) \qquad (9.5)$$

have been calculated and the results are given in table 3:

x_o	y_o	$\overset{1}{M}_{xx}$	$\Delta\overset{1}{M}_{xx}$	$\Delta\overset{1}{M}_{xx}/M_{xx}$
0,5	0,5	0,4275	0,04482	10,48 %
0,5	0,6	0,4542	0,04716	10,38 %
0,5	0,7	0,5264	0,03746	7,11 %
0,5	0,8	0,6268	0,03259	5,20 %
0,5	0,9	0,7237	0,09879	13,65 %
0,6	0,5	0,4306	0,04360	10,13 %
0,7	0,5	0,4400	0,03028	6,88 %
0,8	0,5	0,4534	0,08555	18,87 %
0,9	0,5	0,4725	0,1919	40,62 %
0,6	0,6	0,4578	0,04403	9,62 %
0,7	0,7	0,5358	0,04819	8,99 %
0,8	0,8	0,6425	0,10212	15,89 %
0,9	0,9	0,7573	0,24187	31,94 %

Table 3

REFERENCES

Aronszajn, N.

[1943]: The Rayleigh-Ritz and A. Weinstein methods for approximation of eigenvalues, I. Operators in a Hilbert space, II. Differential operators, Proc. Nat. Acad. Sci. USA 34, 474 and 594

[1951]: Approximation methods for eigenvalues of completely continuous symmetric operators, Proc. Symp. Spectr. Theory and Diff. Prob., Stillwater Okla. Res. Found., 179

Bazley, N.W.

[1961]: Lower bounds for eigenvalues, J. Math. Mech. 10, 289

Bazley, N.W. and Fox, D.W.

[1961]: Truncations in the method of intermediate problems for lower bounds to eigenvalues, J. Res. N.B.S. 65 B. 105

[1962 a]: Lower bounds to eigenvalues using operator decompositions of the form B*B, Arch. Rat. Mech. Anal. 10, 352

[1962 b]: Error bounds for eigenvectors of self-adjoint operators, J. Res. N.B.S. 66 B, 1

Bramble, J.H. and Payne, L.E.

[1963]: Upper and lower bounds in equations of forced vibration type, Arch. Rat. Mech. Anal. 14, 153

Chang, T.H. and Craig, R.R.

[1973]: Computation of upper and lower bounds to frequencies of elastic systems by the method of Lehmann and Maehly, Int. J. Num. Math. in Eng. 6, 323

Courant, R.

[1920]: Über die Eigenwerte bei den Differentialgleichungen der mathematischen Physik, Math. Z. 1

Fichera, G.

[1966]: Approximation and estimates for eigenvalues, Proc. Conf. Num. Anal., Univ. Maryland 1965. In: Num. Sol. Part. Diff. Eqn., Acad. Press New York, 317

Gold, S.H.

[1957]: Variational methods for eigenvalue problems, Univ. Toronto Press, Toronto, 275 p.

Kato, T.

[1953]: On some approximation methods concerning the operators T*T, Math. Ann. 126, 253

Marangoni, R.D., Cook, L.M. and Basavanhally, N.

[1978]: Upper and lower bounds to the natural frequencies of vibration of clamped rectangular orthotropic plates, Int. J. Solids. Struct. 14, 611

Mikhlin, S.G. and Smolitzkiy, K.L.

[1967]: Approximate methods for solution of differential and integral equations, Amer. Elsevier Publ. Comp. New York, 308 p.

Moler, C.B. and Payne, L.E.

[1968]: Bounds for eigenvalues and eigenvectors of symmetric operators, SIAM J. Num. Anal. 5/1, 64

Rieder, G.

[1968]: Zur punktweisen Eingrenzung von Eigenfunktionen in Schwingungs- und Stabilitätsproblemen, ZAMM 48/8, T 279

[1972]: Eingrenzungen in der Elastizitäts- und Potentialtheorie, ZAMM 52, T 340

Stumpf, H.

[1970]: Eingrenzungsverfahren in der Elastomechanik, Forschungsber. Land NRW Nr. 2116, Westdeutsch. Verlag Köln, 105 p.

Stumpf, H.

[1972 a]: Lower bounds to frequencies of continuous elastic systems. In: Continuum mechanics and related problems of analysis, Jubilee-Volume to the 80th birthday of N.I. Muskhelishvili, Moscow, 667

[1972 b]: Zur Eingrenzung von Feldgrößen bei Eigenschwingungsproblemen der Elastomechanik, Acta Mech. 13, 225

[1972 c]: Zur numerischen Berechnung zweiseitiger Schranken für beliebige vektorielle und tensorielle Feldgrößen elastischer Eigenschwingungszuständen, ZAMM 52/1, 37

Stumpf, H. and Biehl, F.J.

[1977]: Die Methode der orthogonalen Projektionen und ihre Anwendung zur Berechnung orthotroper Platten, Mitt. Inst. Mech. 7, Ruhr-Univ. Bochum, FRG

Weinberger, H.F.

[1960]: Error bounds in the Rayleigh-Ritz approximation of eigenvectors, J. Res. N.B.S. B 64, 217

Weinstein, A. and Stenger, W.

[1972]: Methods of intermediate problems for eigenvalues, Acad. Press New York

Trefftz, E.

[1933]: Über Fehlerschätzung bei Berechnung von Eigenwerten, Math. Anal. 108, 595

Stein, P.

[1959]: Die Singularitätenmethode zur Berechnung orthogonal anisotroper Rechteckplatten einschließlich Trägersorten, Stahlbau-Verlag, Köln, 12

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