RUHR-UNIVERSITÄT BOCHUM

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Infinitesimal Thermoplasticity and Kinematics of Finite Elastic-Plastic Deformations. Basic Concepts

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SUMMARY

The first three parts of the paper are devoted to the discussion of the theory of elastic-plastic materials within the framework of classical thermodynamics of irreversible processes. The strains are assumed to be infinitesimal. The notion of hypothetical homogeneous reversible process is introduced, and concepts of constrained equilibrium and local state are used, according to the idea of Kestin and Rice [6]. The notion of the entropy is introduced in classical manner, basing on the Caratheodory's theorem. It is assumed that dissipative forces posses the potential, and with the help of this postulate the generalized non-isothermal plastic flow rules are derived. The possible coupled effects between elastic behaviour, plastic flow and temperature are discussed in detail, and a few alternative forms of equations for the temperature are derived. It is shown how the experimental facts concerning the so called "stored energy due to plastic deformations of metals" could be incorporated into the presented framework. It is noted that due to our inability to contrive a real process which would be close to the assumed hypothetical reversible process it is impossible to separate the reversible heat of internal changes from that due to the dissipation of the mechanical work.

Part IV contains a presentation and comparison of various original ideas contained in the current literature on kinematics of finite elastic-plastic deformations. The basic kinematic quantities like strain and stretching as well as their decompositions into elastic and plastic parts are discussed. The elastic-plastic body is considered as a material with an "intermediate configuration". Relations between the coordinate - free or absolute notation and a representation in convected coordinates are worked out in detail.

ZUSAMMENFASSUNG

Teil I bis III der Arbeit sind einer Diskussion der Theorie elastisch-plastischer Materialien im Rahmen der klassischen Thermodynamik irreversibler Prozesse gewidmet. Die Verzerrungen werden als infinitesimal angenommen. Der Begriff eines hypothetischen, homogenen reversiblen Prozesses wird eingeführt, und es wird, Kestin und Rice [6] folgend, das Konzept des eingeschränkten Gleichgewichts und des lokalen Zustandes benutzt. Der Begriff der Entropie wird in klassischer Form auf der Grundlage Caratheodorys Theorem eingeführt. Es wird angenommen, daß dissipative Kräfte ein Potential besitzen. Mit Hilfe dieses Postulats werden verallgemeinerte nicht isotherme plastische Fließregeln abgeleitet. Die möglichen Kopplungseffekte zwischen elastischem Verhalten, plastischem Fließen und der Temperatur werden im Detail diskutiert und zwei alternative Formen der Gleichungen für die Temperatur abgeleitet. Es wird gezeigt, wie die experimentellen Befunde, die die sogenannte "gespeicherte Energie infolge plastischer Deformationen von Metallen"betreffen, in den dargestellten Rahmen eingearbeitet werden können. Es wird angemerkt, daß es unmöglich ist, die reversible Wärme infolge innerer Änderungen von der durch Dissipation mechanischer Arbeit erzeugten Wärme zu trennen, da wir nicht in der Lage sind, einen wirklichen Prozeß anzugeben, der dem angenommenen hypothetischen reversiblen Prozeß nahekommt.

Teil IV enthält eine Darstellung und einen Vergleich verschiedener aus der Literatur bekannter Überlegungen zur Kinematik endlicher elasto-plastischer Deformationen. Die grundlegenden kinematischen Größen, wie Verzerrungen und Verzerrungsgeschwindigkeiten, und ihre Zerlegungen in elastische und plastische Anteile werden diskutiert. Der elasto-plastische Körper wird als ein Material mit einer "Zwischenkonfiguration" betrachtet. Die Beziehungen zwischen einer koordinatenfreien oder absoluten Schreibweise und der Darstellung in konvektiven Koordinaten werden eingehend besprochen.

PREFACE

These lectures were presented in a seminar for doctor students, organized by Lehrstuhl I für Mechanik at Ruhr-Universität Bochum during winter term 1977/78. They are partly a result of our cooperation and discussions when the first author (B. R.) stayed at Bochum with a fellowship from Alexander von Humboldt-Stiftung. The financial support of this foundation as well as the stimulating discussions with Professor Th. Lehmann are greatfully acknowledged.

Part I to III, dealing with infinitesimal thermoplasticity is based on B. R.'s previous work in the field and has been written by him alone. Part IV gives an introduction to the kinematics of finite elastic-plastic deformations and should be considered as the first step in an attempt to generalize the concepts of infinitesimal thermoplasticity.

Bogdan Raniecki Klaus Thermann

NOTATION

$$\underset{\sim}{A} \cdot \underset{\sim}{B} \longrightarrow A_i B_i \text{ or } A_{ij} B_{ij}$$

$$\underset{\sim}{A}\, \underset{\sim}{B} \longrightarrow A_{ij} B_j \text{ or } A_{ijmn} B_{mn}$$

$$tr\, \underset{\sim}{A} \longrightarrow A_{ii}$$

$$\underset{\sim}{1} \longrightarrow \delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

$$dev\, \underset{\sim}{A} = \underset{\sim}{A} - \frac{1}{3}\, tr(\underset{\sim}{A})\, \underset{\sim}{1}$$

$$\underset{\sim}{L} = \frac{\partial \underset{\sim}{A}}{\partial \underset{\sim}{B}} \longrightarrow L_{ijmn} = \frac{\partial A_{ij}}{\partial B_{mn}}$$

$$(\underset{\sim}{A}^{T})_{ij} = (\underset{\sim}{A})_{ji}$$

$$grad\, \underset{\sim}{A} = \frac{\partial \underset{\sim}{A}}{\partial x_i}$$

$$div\, \underset{\sim}{A} \longrightarrow A_{ij,j}$$

$$(\underset{\sim}{A} \otimes \underset{\sim}{B})_{ij} \longrightarrow A_i B_j$$

$$(\underset{\sim}{A} \otimes \underset{\sim}{B})_{ijmn} \longrightarrow A_{ij} B_{mn}$$

CONTENTS

I HOMOGENEOUS PROCESSES IN UNIFORM MATERIAL SYSTEMS

1. Fundamental Equation of State - Dissipation

1.1. Basic Concepts of Classical Thermodynamics Concerning the First Law

Consider the piece of metal of unit mass whose physical properties (i. e. state variables) in any equilibrium state are the same at every particle. Such a piece of metal is said to be a uniform material system. Assume that the material system is a closed one, i. e., it does not exchange mass with its surrounding, and that it is chemically inert. It is evident that the actual macroscopically homogeneous solid may be regarded to be uniform only for a restricted class of thermomechanical interactions. However, it is this class for which the thermomechanical properties are supposed to be determined in an experimental program. Further development of the theory (part II) which includes non-uniform continuous systems will enable to predict the surface response of a solid under a general class of thermo-mechanical interactions, i. e. it will enable to verify the complete theory.

Let us first briefly recall the most fundamental concepts of thermodynamics [1] concerning the first law. One introduces as primitive notions various idealized walls or partitions such as "adiabatic", "diathermal", "rigid" etc. Such walls are regarded as external constraints which hold the system in equilibrium. The set of independent measurable properties which are relevant for the analysis of a system when the latter is in equilibrium is then called its state. The existence of a function of state called "empirical temperature" θ one deduces from the "Zeroth Law" according to which "if each of two systems is in equilibrium with a third one, then they are in equilibrium with each other". The empirical temperature is then regarded as an independent state variable and together with some number of variables of "displacement"-type describes the state of a system [1].

Denote symbolically by "1" and "2" two equilibrium states of a system. The First Law of Thermodynamics states that:

a) If "1" and "2" are arbitrary prescribed equilibrium states of an adiabatically enclosed system such that a real process from "2" to "1" is impossible then there exists the real process from "1" to "2".

b) The work w done on the system in a real adiabatic process depends on these terminal states alone, i. e., it is independent of the manner in which the process proceeds.

The first law associates with a system the new state function u called internal energy function. In passing we note that in thermodynamics the notion of a state is conceptually related with the notion of the internal energy. Now, the difference between the values of u calculated for any two states is equal to the work done on system in any adiabatic process between these states

$$\Delta u - w(1;2) = 0 \quad ; \quad \Delta u = u(2) - u(1) \tag{1.1}$$

where w (1, 2) is the work done on the system in the transition 1 ⟶ 2. One assumes that the internal energy function is continuous and differentiable. It is additive*)

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*) This is true if effects of surface tension and mutual long-range interaction between different parts of the system are negligible.

i. e. the energy of a compound system is equal to the sum of internal energies of the constitutive system.

Assume that the adiabatic isolation is removed and consider a non-adiabatic process between the same equilibrium states 1 and 2. The left-hand side of equation $(1.1)_1$ will now fail to vanish. It is equal to heat q absorbed by the system in this process

$$q(1,2) = \Delta U - w(1,2) \tag{1.2}$$

The above equation may be regarded as the definition of heat "heat absorbed by a system in any particular process between given states is the difference between the work which would have done on the system had the process between the given terminal states been adiabatic, and the work actually done on a system in this process".

1.2. State of Elastic-Plastic Material System

To describe some observed phenomena the notions of constraints are used also in a more general sense (see e. g. [2]). It is believed that equilibrium is maintained not only by external constraints but also by internal one. The internal constraints may be developed by the system itself in the course of a process, and they do not need to be of concrete nature, like walls or partitions*). One also introduce the associated notion of "constrained equilibrium". It is this notion which was used by Vakulenko [3, 4], Kestin [5], Kestin and Rice [6], when applying the concepts of classical thermodynamics to plastically strained material systems. It was also adopted tacitly by many other writers (see e. g. Lehmann [7, 8]).

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*) For example "a stoichiometric mixture of oxygen and hydrogen at low tempreature reacts to form water vapor at such a slow rate that we can regard the system as having been constrained by an ideal anticatalyst (Kestin [5])".

We assume that the state of an elastic-plastic uniform material system in constrained equilibrium is described by the empirical temperature θ , the tensor of elastic infinitesimal strains $\underset{\sim}{\varepsilon}^{(e)}$, one scalar b and one symmetric second-order tensor $\underset{\sim}{\beta}$

$$\theta,\ \underset{\sim}{\varepsilon}^{(e)},\ b,\ \underset{\sim}{\beta}=\underset{\sim}{\beta}^{T} \tag{1.3}$$

Here we may note:

Remark

1. θ and total strain $\underset{\sim}{\varepsilon}$ alone can not be taken as state variables since the energy function would not be a single-valued function of state.

2. θ, $\underset{\sim}{\varepsilon}^{(e)}$ and the tensor $\underset{\sim}{\varepsilon}^{(i)}$ of inelastic strain (or plastic strain ; $\underset{\sim}{\varepsilon}=\underset{\sim}{\varepsilon}^{(e)}+\underset{\sim}{\varepsilon}^{(i)}$) alone are not appropriate state variables since after an isothermal cycle in $\underset{\sim}{\varepsilon}^{(e)}$, $\underset{\sim}{\varepsilon}^{(i)}$ space we would get from equation (1.2) $q=-w_{i}(\Delta u=0)$ what is not in agreement with experimental observations.

3. The necessity of introductions of quantities such as b and $\underset{\sim}{\beta}$ follows from the following macroscopic observations (i) the heat output in a cyclic isothermal process in $\underset{\sim}{\sigma}-\theta$ space ($\underset{\sim}{\sigma}$ being the macroscopic stress tensor) is not equal to the work done on a material system in such a process. (ii) expansion and translation of the yield surface in $\underset{\sim}{\sigma}$ -space (hardening of metals).

The quantities b and $\underset{\sim}{\beta}$ in a certain global sense represent the change of internal constraints. They are also called internal variables or internal (hidden) coordinates. Viewed macroscopically the internal coordinates are to signify the most dominant aspects of the microstructural changes. Generally, there exist still a lot of uncertainities concerning the number and clear physical meaning of such parameters. Hence, it is still an open question of thermoplasticity what are the operational definitions of internal parameters. Here we intend to describe the observed motion of yield surface in $\underset{\sim}{\sigma}$ -space and for this purpose it is most convenient to assume that one scalar and one tensor form a set of appropriate internal parameters.

1.3. Homogeneous Processes - Work

Under prescribed boundary conditions a real process in a uniform system can not be represented, in general, by a curve in the 14-dimensional space*) of state variables (1.3), for it is the matter of experience that in a system undergoing rapid changes the phenomena such as waves, heat transport preclude the uniformity of the system. However, when the changes of external boundary conditions proceed at an infinitesimal rate the system may be assumed to be uniform at any time of a process and to go through a continuous sequence of states. Such processes are called "homogeneous processes" (or pseudostatic processes [1]). The homogeneous process between two terminal constrained equilibrium states 1 and 2 can be represented by a line in the representative space. For a homogeneous process from a given state to a sufficiently close neighboring state ("infinitesimal homogeneous process") the first law (1.2) can be written as

$$dq = du - dw \tag{1.4}$$

Since the empirical temperature is defined in thermal equilibrium it follows that during the homogeneous process

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*) The 14-dimensional euclidean space with coordinates (1.3) is called thermodynamic configuration space or representative space [1].

the uniform material system is assumed to be not far from the thermal equilibrium with its surroundings. However the external and internal forces which do work in a homogeneous process are not exclusively those which hold the system in a constrained equilibrium. Therefore, in thermodynamics one singles out a special class of homogeneous processes, namely <u>reversible processes</u>*), by the additional requirement that in the course of the latter the work done on a system is done precisely by the forces which hold the system in constrained equilibrium. Such forces are functions of state variables (1.3), and therefore they constitute the additional thermodynamics properties of a system. They are called the <u>reversible thermodynamic forces</u>. One of the most important features of reversible processes is that the heat absorbed by the system and work done on the system are the same for all reversible processes between the given states [2].

Denote by

$$\{\underset{\sim}{\sigma}^{\circ}, \underset{\sim}{\Pi}, \pi\} \qquad (\underset{\sim}{\Pi}^{T} = \underset{\sim}{\Pi}, \ \underset{\sim}{\sigma}^{\circ T} = \underset{\sim}{\sigma}^{\circ})$$

the set of reversible forces corresponding to the set $\{\underset{\sim}{\varepsilon}^{(e)}, \mathcal{L}, b\}$ of external and internal coordinates. In general $\underset{\sim}{\sigma}^{\circ}$ need not to be identified with the macroscopic stress tensor $\underset{\sim}{\sigma}$ acting on a system in real homogeneous process i. e., in general $\underset{\sim}{\sigma}$ is not a function of state variables (1.3) alone. The tensor $\underset{\sim}{\sigma}^{\circ}$ represents the external reversible forces, whereas the forces $\underset{\sim}{\Pi} = \underset{\sim}{\Pi}^{T}$ and π are reversible internal forces. The infinitesimal

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*) The concepts of classical thermodynamics can be applied only to such systems for which one can indicate, at least conceptually, the reversible processes [5].

increment of work in a reversible process w° may be written as

$$d w^\circ = \underset{(R)}{d w} + \underset{(H)}{d w} \tag{1.5}$$

$$\underset{(R)}{d w} = \frac{1}{\rho} \underset{\sim}{\sigma}^\circ \cdot d \underset{\sim}{\varepsilon}^{(e)} ; \quad \underset{(H)}{d w} = \frac{1}{\rho} \pi \, d b + \frac{1}{\rho} \underset{\sim}{\Pi} \cdot d \underset{\sim}{\beta}$$

Here superscript „$\circ$" in w° emphasizes the reversible nature of the process, ρ denotes the density of a material, $\underset{(R)}{w}$ represents recoverable external work. The term $\underset{(H)}{d w}$ represents the work of internal forces (internal work). Macroscopically this work displays in the form of hardening of a metal and can also be detected in the isothermal cyclic process in $\underset{\sim}{\sigma}^\circ$ space. In a reversible process this work could be done by some agency only by means of very special manner of manipulations of the internal constraints. Conceptually it is possible to do so. However, during all real infinitesimal homogeneous processes of plastic deformation some agency performs the work on the system only by direct manipulation of external constraints

$$d w = \frac{1}{\rho} \underset{\sim}{\sigma} \cdot d \underset{\sim}{\varepsilon} = \underbrace{\frac{1}{\rho} \underset{\sim}{\sigma} \cdot d \underset{\sim}{\varepsilon}^{(e)}}_{\underset{(E)}{d w}} + \underbrace{\frac{1}{\rho} \underset{\sim}{\sigma} \cdot d \underset{\sim}{\varepsilon}^{(i)}}_{\underset{(i)}{d w}} \tag{1.6}$$

i. e. in the course of actual homogeneous process internal coordinates b and $\underset{\sim}{\beta}$ may vary though internal forces are unbalanced. Here $\underset{\sim}{\sigma}$ is the macroscopic stress tensor. This tensor is partly balanced i. e. only the part $\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^\circ$ remains unbalanced in an actual homogeneous process. During such process the system may undergo permanent change of shape represented by an increment of inelastic strain $d \underset{\sim}{\varepsilon}^{(i)}$ which is not a state variable. The part $\frac{1}{\rho} \underset{\sim}{\sigma} \cdot d \underset{\sim}{\varepsilon}^{(i)}$

of the total work which does not occur in a reversible process indicate that an actual process is irreversible.

1.4. The Second Law of Thermodynamics - Entropy

a) The Second Law of Thermodynamics which is known as "Principle of Carathéodory" is the following [1]

> "In every neighbourhood of any state 1
> of an adiabatically isolated system there
> are states inaccessible from 1"

Since the Second Law is valid for all adiabatic processes (not necessarily homogeneous or reversible) then it must hold a fortiori for reversible processes. In this section the attenuation will be restricted to the reversible processes. For such processes equation (1.5) holds. Substituting it into (1.4) written for reversible processes, i. e. into $dq^{o}= du - dw^{o}$, and taking into account (1.3) the heat dq^{o} absorbed by a system in an infinitesimal reversible process may be expressed as a linear differential form

$$dq^{o}= \frac{\partial u}{\partial \theta} d\theta + \left(\frac{\partial u}{\partial \underset{\sim}{\varepsilon}^{(e)}} - \frac{1}{\varrho} \underset{\sim}{\sigma}^{o}\right)\cdot d\underset{\sim}{\varepsilon}^{(e)} + \left(\frac{\partial u}{\partial b} - \frac{\pi}{\varrho}\right) db + \left(\frac{\partial u}{\partial \underset{\sim}{\beta}} - \frac{1}{\varrho}\underset{\sim}{\Pi}\right)\cdot d\underset{\sim}{\beta} \quad (1.7)$$

i. e., dq^{o} is a linear homogeneous function of the differentials of 14 independent state variables.

Before we proceed to discuss the conclusions which follow from the Second Law with reference to reversible processes, it appears desirable first to discuss briefly the linear differential forms [1].

b) Consider the linear differential form

$$dL = X_i(x_k)\,dx_i = \sum_{i=1}^{n} X_i(x_k)\,dx_i \tag{1.8}$$

and corresponding total differential equation

$$dL = X_i(x_k)\,dx_i = 0 \tag{1.9}$$

Assume that the functions X_i are continuous and differentiable. It is always possible to find a set of functions $x_i(t)$,$(i = 1,\cdots, n)$ which satisfies the equation

$$X_i(x_k)\,\frac{dx_i}{dt} = 0 \tag{1.10}$$

for if the $n-1$ functions $x_i(t), (i=1,\cdots,n-1)$ are prescribed at will the remaining function $x_n(t)$ may be found by solution of the resulting ordinary differential equation of first order. Such set of functions $x_i(t)$ defines a "solution curve" in n-dimensional Euclidean space E_n with coordinates x_i . However, for particular set of the functions $X_i(x_k)$ there may exist functions $k(x_i)$ and $R(x_i)$ such that the differential form dL can be expressed as

$$dL = k(x_i)\,dR(x_i) = k(x_i)\,\frac{\partial R}{\partial x_j}\,dx_j \tag{1.11}$$

so that

$$X_j = k(x_i)\,\frac{\partial R}{\partial x_j} \tag{1.12}$$

The differential form dL is then called "<u>integrable</u>" since the equation (1.9) reduces to $dR=0$ which in turn has the solution

$$R(x_i) = const \tag{1.13}$$

The function $k(x_i)$ is said to be integrating denominator of dL. Note that k and R are not uniquely defined. For if some pair of such functions k, R has been found then

$$k^* = k \Big/ \frac{df(R)}{dR} \quad ; \quad R^* = f(R)$$

is an equivalent pair, where f is an arbitrary function of R. The necessary and sufficient condition for the differential form dL to be integrable is the following

$$G_{ijk} = X_i \left(\frac{\partial X_j}{\partial x_k} - \frac{\partial X_k}{\partial x_j} \right) + X_j \left(\frac{\partial X_k}{\partial x_i} - \frac{\partial X_i}{\partial x_k} \right) + X_k \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \quad (1.14)$$
$$= 0$$

Note that G_{ijk} is not identically equal to zero only if $i \neq j \neq k$. Hence, when $n = 2$ the differential form is integrable. To check whether the given form dL is integrable or not, the condition (1.14) requires knowledge of explicite forms of functions X_i. In thermodynamics, however, the use is made of a "qualitative" criterion which follows from the fact that when the form dL is integrable any "solution curve" of (1.9) which passes through a prescribed initial point (in E_n) must lie on a fixed hyper-surface R = const. containing this initial point. This criterion is known as Carathéodory's theorem.

"If every neighbourhood of any arbitrary point 1 in E_n contains points 1' inaccessible from 1 along solution curves of the equation (1.9) then the form dL is integrable".

c) Using the above Carathéodory theorem we may conclude form the Second Law of Thermodynamics that differential form (1.7) for dq° is integrable. There exist functions, we denote by T and s , such that

$$dq^{\circ} = Tds = du - \frac{1}{\varrho}\underset{\sim}{\sigma}^{\circ}\cdot d\underset{\sim}{\varepsilon}^{(e)} - \frac{1}{\varrho}\underset{\sim}{\Pi}\cdot d\underset{\sim}{\beta} - \frac{1}{\varrho}\pi\, db \tag{1.15}$$

or

$$du = Tds + \frac{1}{\varrho}\underset{\sim}{\sigma}^{\circ}\cdot d\underset{\sim}{\varepsilon}^{(e)} + \frac{1}{\varrho}\underset{\sim}{\Pi}\cdot d\underset{\sim}{\beta} + \frac{1}{\varrho}\pi\, db \tag{1.16}$$

Further conclusions which follow from the Second Law are the following*):

1. The integrating denominator T can be chosen so that it is made to be a unique function of the empirical temperature θ alone for any uniform material system i. e. independently of specific properties of the system concerned

   $$T = T(\theta)$$

   For this reason $T(\theta)$ is called the <u>absolute temperature function</u>. The sign of the absolute temperature is fixed by convention to be positive.

2. The function s corresponding to absolute temperature is called metrical**) <u>entropy function</u>. The entropy is <u>additive</u> i. e. the entropy of a compound material system consisting of two uniform material subsystems is the sum of the entropies of its constituent systems.

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*) The justification is rather lengthy but concerns any thermodynamical system. It can be found in the book [1].

**) The word "metrical" is omitted in further analysis.

Both T and s are functions of state of a material system. Therefore Θ can be eliminated from the equations

$$s(\underset{\sim}{\varepsilon}^{(e)}, \Theta, \underset{\sim}{\beta}, b)\ ; \qquad T = T(\Theta)$$

and we can use the set

$$Y^{s\varepsilon} = \{s, \underset{\sim}{\varepsilon}^{(e)}, \underset{\sim}{\beta}, b\} \tag{1.17}$$

as the set of independent properties constituting the state of the material system

$$du(Y^{s\varepsilon}) = T(Y^{s\varepsilon})\,ds + \frac{1}{\rho}\underset{\sim}{\sigma}^{o}(Y^{s\varepsilon})\cdot d\underset{\sim}{\varepsilon}^{(e)} + \frac{1}{\rho}\underset{\sim}{\Pi}(Y^{s\varepsilon})\cdot d\underset{\sim}{\beta} + \frac{1}{\rho}\pi(Y^{s\varepsilon})\,db \tag{1.18}$$

From (1.18) we get the set of so called "<u>state equations</u>"

$$\begin{aligned} T &= \frac{\partial u(Y^{s\varepsilon})}{\partial s}\ ; & \frac{1}{\rho}\underset{\sim}{\Pi} &= \frac{\partial u(Y^{s\varepsilon})}{\partial \underset{\sim}{\beta}}\ ; \\ \frac{1}{\rho}\underset{\sim}{\sigma}^{o} &= \frac{\partial u(Y^{s\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}}\ ; & \frac{1}{\rho}\pi &= \frac{\partial u(Y^{s\varepsilon})}{\partial b} \end{aligned} \tag{1.19}$$

Once the internal energy is a known function of variables $Y^{s\varepsilon}$ all equilibrium properties of a material system described by (1.19) are known. For this reason equation (1.18) is called <u>Gibbs fundamental equation of state</u>, and the function $u(Y^{s\varepsilon})$ - the <u>thermodynamic potential</u>. Note, however, that the internal energy is a thermodynamic potential only if it is expressed in terms of variables $Y^{s\varepsilon}$ i. e. for example $u(T, \underset{\sim}{\varepsilon}^{(e)}, \underset{\sim}{\beta}, b)$ or $u(s, \underset{\sim}{\sigma}^{o}, \underset{\sim}{\beta}, b)$ are functions of state but they are not thermodynamic potentials. Note that from $(1.15)_1$ it follows that in the course of any reversible adiabatic process $(dq^{o} = 0)$ we have $ds = 0$, i. e. a reversible adiabatic process coincides with the isentropic one.

1.5. Thermodynamic Potentials

The functions which are derived by application of the technique of Legendre transformation (Appendix A) to the internal energy function $u(Y^{s\varepsilon})$ are called <u>thermodynamic potentials</u>. Knowledge of a single such function is equivalent to knowledge of all equations of state which in turn contain complete information on the thermostatic properties of a material system. The different Legendre transforms of u have different sets of independent variables which can be chosen at will.

The stress, strain and the temperature are the physical quantities which can easier be controled and measured than entropy in any experimental programm. It is in applications more convenient to use the following two sets

$$Y^{T\varepsilon} = \left\{ T, \underset{\sim}{\varepsilon}^{(e)}, \underset{\sim}{\beta}, b \right\} \quad \text{or} \quad Y^{T\sigma} = \left\{ T, \underset{\sim}{\sigma}^{\circ}, \underset{\sim}{\beta}, b \right\} \qquad (1.20)$$

as sets of independent variables, rather then the sets used hitherto.

The Helmholtz potential or Helmholtz <u>free energy</u> φ, when taken with reverse sign, is that partial Legendre transform of u which replaces the entropy by the temperature as independent variable. The natural variables of the free energy function are the elements of $Y^{T\varepsilon}$, thus

$$\varphi(Y^{T\varepsilon}) = \left[u - Ts \right]_{s=s(Y^{T\varepsilon})} \qquad (1.21)$$

Alternatively, if the set $Y^{T\sigma}$ is taken as a set of independent variables then the so called <u>Gibbs function</u> or Gibbs free energy is the appriopriate thermodynamical potential. The Gibbs function ϱg, while taken with reverse sign is the Legendre transform of $\varrho\varphi$ with respect to variables $\underset{\sim}{\varepsilon}^{(e)}$,

$$\rho\, g(Y^{T\sigma}) = \left[\rho\,\varphi - \underset{\sim}{\sigma}^{\circ}\cdot\underset{\sim}{\varepsilon}^{(e)}\right]_{\underset{\sim}{\varepsilon}^{(e)}=\underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma})} \qquad (1.22)$$

The complete differentials $d\varphi$ and dg are

$$d\varphi = du - Tds - sdT = -sdT + \frac{1}{\rho}\underset{\sim}{\sigma}^{\circ}\cdot d\underset{\sim}{\varepsilon}^{(e)} + \frac{1}{\rho}\underset{\sim}{\Pi}\cdot d\underset{\sim}{\beta} + \frac{\pi}{\rho}db \qquad (1.23)$$

$$dg = -sdT - \frac{1}{\rho}\underset{\sim}{\varepsilon}^{(e)}\cdot d\underset{\sim}{\sigma}^{\circ} + \frac{1}{\rho}\underset{\sim}{\Pi}\cdot d\underset{\sim}{\beta} + \frac{1}{\rho}\pi\, db \qquad (1.24)$$

on account of (1.18), (1.21) and (1.22).

The full relationship between internal energy, free energy and Gibbs function is summarized in the following schematic comparison*).

Internal energy	Helmholtz free energy	Gibbs function	
$u = u(Y^{s\varepsilon})$	$\varphi(Y^{T\varepsilon}) = u - Ts$	$g(Y^{T\sigma}) = \varphi - \frac{1}{\rho}\underset{\sim}{\sigma}^{\circ}\cdot\underset{\sim}{\varepsilon}^{(e)}$	
$T = \frac{\partial u(Y^{s\varepsilon})}{\partial s}$	$s = -\frac{\partial\varphi(Y^{T\varepsilon})}{\partial T}$	$s = -\frac{\partial g(Y^{T\sigma})}{\partial T}$	(1.25)
$\frac{1}{\rho}\underset{\sim}{\sigma}^{\circ} = \frac{\partial u(Y^{s\varepsilon})}{\partial\underset{\sim}{\varepsilon}^{(e)}}$	$\frac{1}{\rho}\underset{\sim}{\sigma}^{\circ} = \frac{\partial\varphi(Y^{T\varepsilon})}{\partial\underset{\sim}{\varepsilon}^{(e)}}$	$\frac{1}{\rho}\underset{\sim}{\varepsilon}^{(e)} = -\frac{\partial g(Y^{T\sigma})}{\partial\underset{\sim}{\sigma}^{\circ}}$	(1.26)
$\frac{1}{\rho}\underset{\sim}{\Pi} = \frac{\partial u(Y^{s\varepsilon})}{\partial\underset{\sim}{\beta}}$	$\frac{1}{\rho}\underset{\sim}{\Pi} = \frac{\partial\varphi(Y^{T\varepsilon})}{\partial\underset{\sim}{\beta}}$	$\frac{1}{\rho}\underset{\sim}{\Pi} = \frac{\partial g(Y^{T\sigma})}{\partial\underset{\sim}{\beta}}$	(1.27)
$\frac{1}{\rho}\pi = \frac{\partial u(Y^{s\varepsilon})}{\partial b}$	$\frac{1}{\rho}\pi = \frac{\partial\varphi(Y^{T\varepsilon})}{\partial b}$	$\frac{1}{\rho}\pi = \frac{\partial g(Y^{T\sigma})}{\partial b}$	(1.28)

Table 1

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*) Equations (1.25) and (1.26) are sometimes called "caloric equations of state" and "thermic equations of state" respectively [9].

1.6. The Entropy Principle - Energy Dissipation

a) So far in section 1.4 only part of the content of the Second Law has been exhausted. Namely, the entropy S and absolute temperature T have been defined in the course of arguments relating to reversible processes alone. The statement of the Second Law, however, concerns adiabatic processes of any kind. Therefore, the further conclusion may be drawn from it, which is referred to as "<u>Entropy Principle</u>":*)

"The entropy of the final equilibrium state 2 of any adiabatic process is never less than that of its initial state 1".

$$\Delta S = S_2 - S_1 \geqslant 0 \tag{1.29}$$

b) Consider an arbitrary (non-adiabatic) homogeneous process in a material system A (Fig. 1) between given states 1 and 2. To account for the interactions of A with its surroundings assume the latter to be equivalent to an ideal system B [2] in perfect thermal contact with A , the compound system C being adiabatically isolated (Fig. 1). By an ideal system B we mean a system which undergoes only reversible processes.

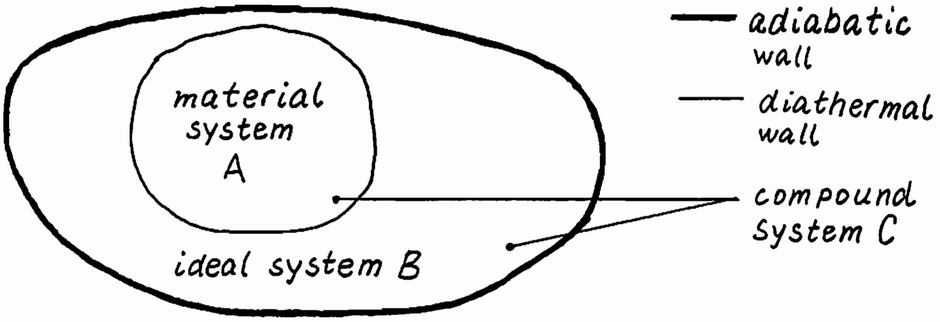

Fig. 1

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*) When the Second Law is formulated in the form given in sec. 1.4 then the "Entropy Principle" can be deduced from it only under the assumption that the set of states accessible in adiabatic processes from a given state is a connected set [1]. Therefore, the Entropy Principle is often referred to as "Second Part of the Second Law of Thermodynamics" [5].

Applying the entropy principle to the compound system C we get

$$\Delta s^{(A)} + \Delta s^{(B)} \geqslant 0$$

But

$$\Delta s^{(B)} = \int_1^2 \frac{dq^{o(B)}}{T^{(B)}}$$

and

$$dq^{o(B)} = -dq^{(A)}; \quad T^{(A)} = T^{(B)}$$

on account of diathermic contact.

Thus, for the system A we may write (superscript (A)) is henceforth omitted)

$$\Delta s - \int_1^2 \frac{dq(t)}{T(t)} \geqslant 0 \qquad (1.30)$$

This is so called "<u>Inequality of Clausius</u>". Its counterpart for infinitesimal homogeneous processes is the following

$$T ds - dq \geqslant 0 \qquad (1.31)$$

c) Consider two sufficiently close neighboring states 1 and 1'. For an infinitesimal homogeneous process 1 ⟶ 1' the equation (1.4) is valid, and dq and dw can be measured. Let us compare these quantities with the corresponding quantities of heat dq^o and work dw^o which are exchanged in the course of an infinitesimal reversible process between the same states. dq^o and dw^o are uniquely determined according to the properties of reversible processes ($dq^o = T ds$ and the increment of the entropy does not depend on the manner in which the infinitesimal process proceeds). However, in general

$$dq^{o} \neq dq \qquad \text{and} \qquad dw^{o} \neq dw$$

but the change of internal energy du is the same. Thus,

$$du = dq^{o} + dw^{o} = dq + dw$$

or

$$\underset{(D)}{dw} \equiv dw - dw^{o} = dq^{o} - dq \tag{1.32}$$

where $\underset{(D)}{dw}$ is so called "energy dissipation" or "work dissipation" in an infinitesimal homogeneous process.

Substituting (1.5) and (1.6) into (1.32), and accounting for that $dq^{o} = Tds$ we get

$$Tds = dq + \underset{(D)}{dw} \tag{1.33}$$

$$\begin{aligned} T\underset{(i)}{ds} &\equiv \underset{(D)}{dw} = dw - dw^{o} \\ &= dw - \underset{(R)}{dw} - \underset{(H)}{dw} = \underset{(E)}{dw} + \underset{(i)}{dw} - \underset{(R)}{dw} - \underset{(H)}{dw} \geqslant 0 \end{aligned} \tag{1.34}$$

or

$$\begin{aligned} \varrho T\underset{(i)}{ds} &= \underset{\sim}{\sigma}^{d} \cdot d\underset{\sim}{\varepsilon}^{(e)} + \underset{\sim}{\sigma} \cdot d\underset{\sim}{\varepsilon}^{(i)} - \underset{\sim}{\Pi} \cdot d\underset{\sim}{\beta} - \pi\, db \geqslant 0 \\ \underset{\sim}{\sigma}^{d} &= \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{o} \end{aligned} \tag{1.35}$$

on account of (1.31). Here $\underset{\sim}{\sigma}^{d} = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{o}$ is so called dissipative stress tensor, and $\underset{(i)}{ds}$ is the "entropy production" due to dissipation in an infinitesimal homogeneous process. The set $\{\underset{\sim}{\sigma}^{d}, -\underset{\sim}{\Pi}, -\pi\}$ of forces occuring in (1.35) will be called the set of dissipative thermodynamics forces. Note that from (1.32) and (1.34) it follows one of the most important properties of reversible processes, namely, of all homogeneous processes occuring between a given initial and a given final state of a uniform material system, the amount of heat $-dq^{o}$ delivered by the system to surroundings and the work done on a system are minima for <u>reversible processes</u>.

2. The Fundamental Postulate - Basic Problems of Thermoplasticity for Homogeneous Processes

a) The notions of empirical temperature, entropy and absolute temperature are defined in the course of arguments relating to reversible processes and, therefore, equations of state may be strictly used only when analysing such ideal processes. To go beyond the usual "thermostatics" and to be able to analyse arbitrary homogeneous process we assume the following postulate

> "We assume that the fundamental equation of state (1.18) is valid in the course of any homogeneous process".

This is the particular form of the more general postulate of local state formulated in part II. Equation (1.33) and (1.34) can be rewritten introducing rates

$$T\dot{s} = \frac{1}{\varrho}\left(\underset{\sim}{\sigma}^{d}\cdot\dot{\underset{\sim}{\varepsilon}}^{(e)} + \underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{\varepsilon}}^{(i)} - \underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}} - \pi\dot{b}\right) + \dot{q} \tag{2.1}$$

$$\varrho T\underset{(i)}{\dot{s}} = \underset{\sim}{\sigma}^{d}\cdot\dot{\underset{\sim}{\varepsilon}}^{(e)} + \sigma\cdot\dot{\underset{\sim}{\varepsilon}}^{(i)} - \underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}} - \pi\dot{b} = \varrho\underset{(D)}{\dot{W}} \geqslant 0 \tag{2.2}$$

b) The basic problem of thermoplasticity for homogeneous processes may be formulated in a few alternative forms. Generally one can prescribe arbitrarily 7 functions

e. g. one of the following pairs*),

$$\{T, \underset{\sim}{\sigma}\}\ ;\ \{T, \underset{\sim}{\varepsilon}\}\ ;\ \{q, \underset{\sim}{\sigma}\}\ ;\ \{q, \underset{\sim}{\varepsilon}\}$$

and assume that the initial $(t=t_0)$ internal state is known

$$\underset{\sim}{\beta}\Big|_{t=t_0} = \underset{\sim}{\beta}_0\ ; \qquad b\Big|_{t=t_0} = b_0 \qquad (2.3)$$

The wish is to determine all other quantities representing the thermodynamics properties of a uniform material system.

Consider one of such problems.

Problem

Given 7 functions

$$T(t)\ ;\ \underset{\sim}{\sigma}(t)\ ;\ t \in [t_0, t_1]$$

and the initial values (2.3). Find 47 functions

$$\underset{\sim}{\varepsilon}(t),\ \underset{\sim}{\varepsilon}^{(e)}(t),\ \underset{\sim}{\varepsilon}^{(i)}(t),\ \underset{\sim}{\sigma}^{d}(t),\ \underset{\sim}{\sigma}^{o}(t),\ \underset{\sim}{\beta}(t),\ \underset{\sim}{\Pi}(t),\ u(t),\ s(t),\ b(t),\ \pi(t),\ q(t)$$

which satisfy (2.3).

The above problem can be solved by using the following 47 equations:

1. $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^{(e)} + \underset{\sim}{\varepsilon}^{(i)}$, $\underset{\sim}{\sigma}^{d} = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{o}$ and the equation (2.1) which follows from the First Law and the first part of the Second Law of thermodynamics.
2. 34 "constitutive equations" which describe the properties of the particular material. These are expressed

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*) Th. Lehmann [7, 9] introduces the term "independent process variables" for such a pair.

through two groups of statements:

i.) One internal energy function u and 14 equations of state which are derivable from u *) (equations contained in the first column of table 1). This group of equations describes the thermostatic properties of the material.

i.i.) 19 rate equations (or "phenomenological equations") which link the dissipative forces $\{\underset{\sim}{\sigma}^{d}, \underset{\sim}{\sigma}, -\underset{\sim}{\Pi}, -\pi\}$ with rates $\{\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}\}$ occuring in the expression (2.2) for the entropy production. The general form of such equations are very briefly discussed in sec. 3.

3. Rate Equations

a) There is no generally acceptable principle to follow when postulating the rate equation. We assume, therefore, that they have the following general form

$$\underset{\sim}{\sigma}^{d} = \underset{\sim}{\psi}_{0}\,(\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}, Y_{b}^{T}) \tag{3.1}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{\psi}_{1}\,(\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}, Y_{b}^{T}) \tag{3.2}$$

$$-\underset{\sim}{\Pi} = \underset{\sim}{\psi}_{2}\,(\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}, Y_{b}^{T}) \tag{3.3}$$

$$-\pi = \psi_{3}\,(\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}, Y_{b}^{T}) \tag{3.4}$$

---

*) Equivalently one can use any other thermodynamic potential and corresponding equations of states (e. g. equations of second or third column in table 1).

where Y_b^T denotes the set consisting of internal variables and temperature

$$Y_b^T = \{ \underset{\sim}{\beta}, b, T \}$$

The functions $\underset{\sim}{\psi}_0, \underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2, \psi_3$ have to be choosen in such a manner that inequality (2.2) is satisfied

$$\rho \underset{(D)}{\dot{w}} = D (\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}, Y_b^T)$$

$$= \underset{\sim}{\psi}_0 \cdot \dot{\underset{\sim}{\varepsilon}}^{(e)} + \underset{\sim}{\psi}_1 \cdot \dot{\underset{\sim}{\varepsilon}}^{(i)} + \underset{\sim}{\psi}_2 \cdot \dot{\underset{\sim}{\beta}} + \psi_3 \dot{b} \geqslant 0 \qquad (3.5)$$

for every admissible process. Here D is called the <u>dissipative function</u>. More specific functions (3.1)-(3.4) can be obtained by assuming that forces $\underset{\sim}{\sigma}^d, \underset{\sim}{\sigma}, -\underset{\sim}{\Pi}, -\pi$ possess so called <u>dissipative potential</u>

$$\psi (\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}, Y_b^T)$$

such that

$$\underset{\sim}{\sigma}^d = \frac{\partial \psi}{\partial \dot{\underset{\sim}{\varepsilon}}^{(e)}} \; ; \quad \underset{\sim}{\sigma} = \frac{\partial \psi}{\partial \dot{\underset{\sim}{\varepsilon}}^{(i)}} \; ; \quad -\underset{\sim}{\Pi} = \frac{\partial \psi}{\partial \dot{\underset{\sim}{\beta}}} \; ; \quad -\pi = \frac{\partial \psi}{\partial \dot{b}} \qquad (3.6)$$

When the potential ψ has continous second derivatives with respect to all rates the assumption (3.6) is equivalent to so called "Gyarmaty postulat" [10]. Recently Edelen [11, 12] has shown that this postulat exclude from the consideration only such forces which when multiplied by conjugate rate are equal to zero i. e. so called "nondissipative forces". Some of the writers (see e. g. [13, 14, 15]) assume additionally that the dissipative potential is a convex function of variables $\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}$.

In the case of materials with instantaneous elasticity range (no visco-elastic effects) it is expedient to assume that

$$\underset{\sim}{\psi}_0 = \underset{\sim}{0} \qquad \left(\text{or } \frac{\partial \psi}{\partial \dot{\underset{\sim}{\varepsilon}}^{(e)}} = \underset{\sim}{0}\right) \tag{3.7}$$

so that (cf. (1.5), (1.6) and (1.34))

$$\underset{\sim}{\sigma}^{d} = \underset{\sim}{0}\,; \qquad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{o} \tag{3.8}$$

$$\dot{W}_{(R)} = \dot{W}_{(E)}\,; \qquad \dot{W}_{(D)} = \dot{W} - \dot{W}_{(E)} - \dot{W}_{(H)} = \dot{W}_{(i)} - \dot{W}_{(H)} \tag{3.9}$$

Henceforth in this section and part III eq. (3.7) will be assumed to be satisfied.

b) If the rate equations (3.2-3.4) are invariant under a change of time scale then the material is said to be "<u>rate independent</u>". Assume the postulate (3.6). For rate independent elastic-plastic materials dissipative potential ψ does not depend on $\dot{\underset{\sim}{\varepsilon}}^{(e)}$ and is a homogeneous function of order one with respect to the variables $\dot{\underset{\sim}{x}}^{D} = \{\dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{k}\}$ i. e.

$$\psi(\tau \dot{\underset{\sim}{x}}^{D}, Y_k^T) = \tau \psi(\dot{\underset{\sim}{x}}^{D}, Y_k^T) \quad \text{for arbitrary } \tau > 0 \tag{3.10}$$

and the surfaces ψ = const. are the same as the surfaces D = const. in $\dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{k}$ space. Moreover the dissipative forces

$$\underset{\sim}{X} = \{\underset{\sim}{\sigma}, -\underset{\sim}{\Pi}, -\pi\} \tag{3.11}$$

are not independent. There exists a function F_1 called "yield function" such that

$$F_1(\underset{\sim}{X}, Y_k^T) = 0 \quad \text{if } \dot{\underset{\sim}{x}}^{D} \neq \underset{\sim}{0} \tag{3.12}$$

and the relations inverse with respect to (3.6) have the form (see appendix A, sec. 2)

$$\dot{\underset{\sim}{x}}^{\mathcal{D}} = \Lambda \frac{\partial F_1}{\partial \underset{\sim}{X}} \quad \text{if} \quad \dot{\underset{\sim}{x}}^{\mathcal{D}} \neq \underset{\sim}{0} \quad (F_1 = 0) \tag{3.13}$$

Inequality (2.2) (with $\underset{\sim}{\sigma}^{d} = \underset{\sim}{0}$) can be satisfied only if

$$\Lambda \geqslant 0 \quad \text{and} \quad \frac{\partial F_1}{\partial \underset{\sim}{\sigma}} \cdot \underset{\sim}{\sigma} + \frac{\partial F_1}{\partial \underset{\sim}{\Pi}} \cdot \underset{\sim}{\Pi} + \frac{\partial F_1}{\partial \pi} > 0 \quad (F_1 = 0) \tag{3.14}$$

Assuming additionally that states with $F_1 > 0$ are inaccessible and adopting an appriopriate "unloading criterion" we get the following <u>generalized nonisothermal plastic flow rules</u>

$$\dot{\underset{\sim}{\varepsilon}}^{(i)} = \Lambda \frac{\partial F_1}{\partial \underset{\sim}{\sigma}}\,; \quad -\dot{\underset{\sim}{\beta}} = \Lambda \frac{\partial F_1}{\partial \underset{\sim}{\Pi}}\,; \quad -\dot{b} = \Lambda \frac{\partial F_1}{\partial \pi} \tag{3.15}$$

$$\text{if} \quad F_1 = \dot{F}_1 = 0\,, \ \Lambda \geqslant 0$$

$$\dot{\underset{\sim}{\varepsilon}}^{(i)} = \underset{\sim}{0}\,; \quad \dot{\underset{\sim}{\beta}} = \underset{\sim}{0}\,; \quad \dot{b} = 0 \tag{3.16}$$

$$\text{if} \quad F_1 < 0 \quad \text{or} \quad F_1 = 0 \ \text{and} \ \overset{\nabla}{F}_1 < 0$$

where

$$\overset{\nabla}{F}_1 = \frac{d}{dt}\left\{ F_1 \Big|_{\substack{\underset{\sim}{\beta} = const \\ b = const}} \right\} \tag{3.17}$$

Here $\underset{\sim}{\Pi} = \underset{\sim}{\Pi}(\underset{\sim}{\sigma}, \underset{\sim}{\beta}, b, T)$ and $\pi = \pi(\underset{\sim}{\sigma}, \underset{\sim}{\beta}, b, T)$ are state functions related with the Gibbs function g by $(1.27)_3$ and $(1.28)_3$ $(\underset{\sim}{\sigma}^{o} = \underset{\sim}{\sigma})$. Note that until $\underset{\sim}{\Pi}$ and π are independent of $\underset{\sim}{\sigma}$ the equations (3.15) differs essentially from associated plastic flow rules [16]*).

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*) $\underset{\sim}{\Pi}$ and π are independent of $\underset{\sim}{\sigma}$ if elastic properties are not influenced by plastic deformation (see part III).

Note also that in the presented approximation the thermodynamic potential (for example Gibbs function g) and the function F_1 completely describe the properties of elastic-plastic rate-independent materials [17, 18].

II CONTINUOUS MATERIAL SYSTEMS - BASIC POSTULATES

4. Physical Concepts of a Continuum [5]

In this short part we shall underline very briefly the main concepts which are used when dealing with material systems the thermodynamical properties of which are different in various parts. Since we intend to use the language of continuum physics we imagine that a given piece of metal has been divided into a finite number (say N) of elementary subsystems (cells) of volume ΔV . It is supposed that the volume ΔV is very small at macroscopic scale and at the same time it must contain a large number of crystallites in order the principles of statistics and method of phenomenological thermodynamics to be applicable to them. A reasonable elementary subsystem is supposed to have volume of order not less then 10^{-9} cm^3 what corresponds to about 10^{15} number of atoms. Assume that mass Δm_i and internal energy of every cell can be determined. The mass of the whole composite system is

$$M = \sum_{i=1}^{N} \bar{\varrho}_i \Delta V \qquad (4.1)$$

where $\bar{\varrho}_i = \Delta m_i / \Delta V$ is the average density. Similarly any extensive*) quantity (i. e. quantity which is proportional to mass in a uniform system) representing a property

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*) Temperature and stress are examples of nonextensive quantities which in thermodynamics are termed "intensive quantities".

of the piece of metal may be expressed as a sum

$$Z = \sum_{i=1}^{N} \bar{z}_i \Delta m_i = \sum_{i=1}^{N} \bar{z}_i \bar{\varrho}_i \Delta V \qquad (4.2)$$

where $\bar{z}_i$ is the corresponding <u>specific</u> quantity per unit of mass $\bar{z}_i = z / \Delta m_i$. For example the total internal energy is a sum of specific internal energies

$$U = \sum_{i=1}^{N} \bar{u}_i \bar{\varrho}_i \Delta V \qquad (4.3)$$

When one adopts the continuum point of view, one approximates the sums (4.1)-(4.3) by integrals according to the formula

$$Z = \int_V z(x_i) \varrho(x_i) dV + r_Z \qquad (4.4)$$

where r_Z is the remainder. Bearing in mind the order of magnitude of ΔV one can neglect r_Z asserting that

$$r_Z \ll Z \qquad (4.5)$$

The above equations (4.4)-(4.5) may be used to estimate whether the continuum approach represents an adequate approximation in any particular set of circumstances.

When one utilizes the continuum approach, while examining the behaviour of a thermodynamical system, the latter is termed a "continuous thermodynamical system". The usual notion of a "body" in Mechanics of Continuous Media may be regarded here as a synonym of "continuous thermodynamical system" and a "particle" may be identified with infinitesimal cell provided that inequalities of the type (4.5) are satisfied.

5. Principle of Local State

Let the body A occupy at fixed time the spatial region D with boundary S. With every point of a space we associate an "infinitesimal elementary material system" or "particle". The position of a particle is indicated by radius vector $\vec{r}$ whose components in some Cartesian coordinate system are (Fig. 2)

$$\vec{r} = \{x_i\}$$

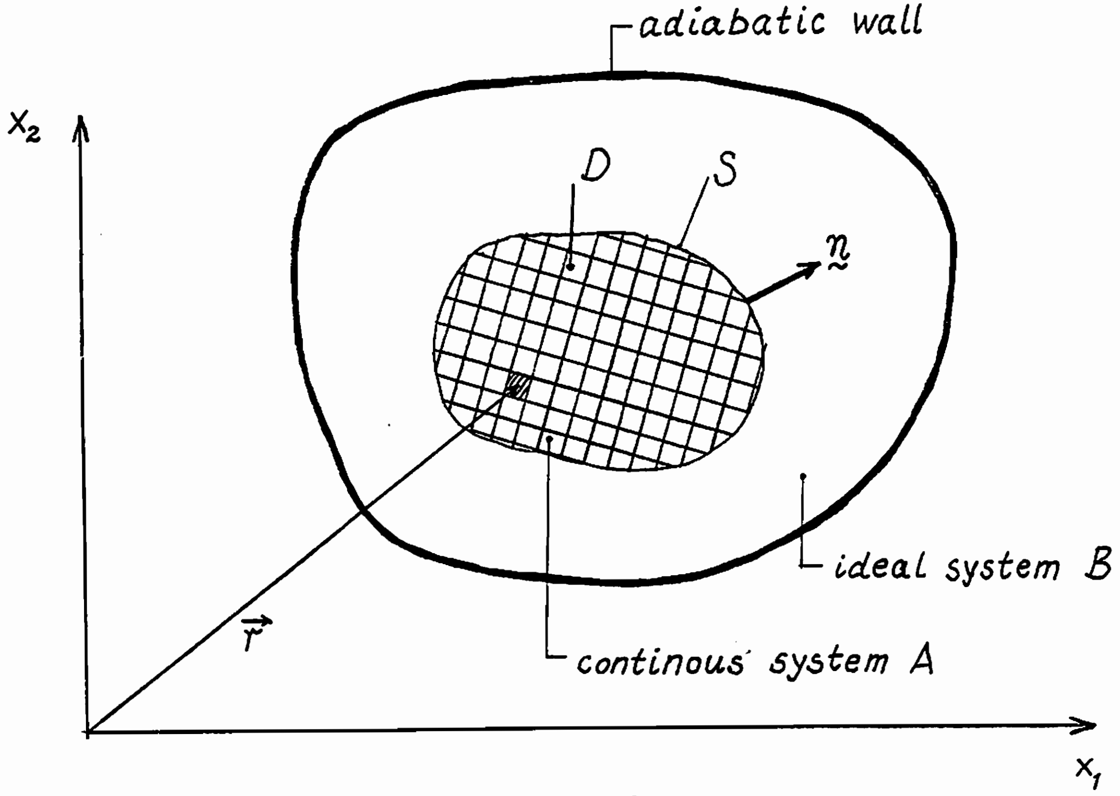

Fig. 2

To analyse processes which take place in continuous material system we shall adopt the approximate formalism which is known as the "classical thermodynamics of irreversible processes" [6, 19, 20]. Accordingly we adopt the <u>principle of local state</u>. It asserts that the continuous system may be thought of as a collection of elementary subsystems each of which undergoes a homogeneous irreversible process and that the fundamental equation of state, hence also all equations of state discussed in the previous sections remain valid locally and instantaneously. In other words, the state of an elementary subsystem is determined by the same thermodynamic properties as would be used if the subsystem were uniform instantaneously. The rates and spatial gradients are specifically excluded from the description of state of a body. In particular, we are free to use the notions of temperature and entropy understanding that these quantities are defined in the manner given in part I. In accordance with the principle of local state, the "nonequilibrium state" of the whole body is described by 14 fields

$$\underset{\sim}{\varepsilon}^{(\theta)}(x_i) \;, \quad s(x_i) \;, \quad \underset{\sim}{\beta}(x_i) \;, \quad b(x_i)$$

and a velocity field $\dot{u}$. Here $s(x_i)$ is the specific entropy field.

The determination of the range of validity of the principle of local state is the subject of separate, both theoretical and experimental studies. In thermoelasticity this problem is partially resolved by justification that the principle can be adopted if noticable changes of strain and temperature occur only in a time period much greater than the so called times of relaxation τ_ε and τ_T , respectively, and if the spatial gradients of strain and temperature are bounded. The values of τ_ε , τ_T depend, of course, on many factors but one can assert that they are of the order

$$\tau_\varepsilon \sim 10^{-9}\ sec \quad , \quad \tau_T \sim 10^{-12}\ sec$$

Thus even the phenomena associated with strain rates of the order 10^7 sec^{-1} could be analysed within this approximate theory. There exists, however, very little informations concerning the times of relaxation related to internal coordinates. Some discussion may be found in recent paper [21].

6. Entropy Production when Heat Conduction is Taken into Account

Let us recall the local form of the principles of conservation of momentum and moment of momentum assuming that there are no acting body forces and couple stresses. These principle are

$$\operatorname{div} \underset{\sim}{\sigma} = \rho \underset{\sim}{\ddot{u}} \qquad \text{and} \qquad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{T} \tag{6.1}$$

respectively, where $\underset{\sim}{u}$ is the displacement vector. Using (6.1) and assuming that there are no heat sources within the system, the local form of the principle of conservation of energy takes the form (see e. g. [22])

$$\dot{u} = \frac{1}{\rho} \underset{\sim}{\sigma} \cdot \underset{\sim}{\dot{\varepsilon}} - \frac{1}{\rho} \operatorname{div} \underset{\sim}{q} \tag{6.2}$$

Here $\underset{\sim}{q}$ is the heat flux per unit time and unit surface, $\underset{\sim}{\varepsilon}$ is the infinitesimal strain tensor related with the displacement vector $\underset{\sim}{u}$ by

$$2\underset{\sim}{\varepsilon} = (\operatorname{grad} \underset{\sim}{u}) + (\operatorname{grad} \underset{\sim}{u})^{T} \tag{6.3}$$

Equation (6.2) corresponds to equations (1.4) and (1.6) taken together i. e. it is sufficient to interpretate $\dot{q}$ in (1.4) as

$$\dot{q} = -\frac{1}{\rho} \operatorname{div} \underset{\sim}{q} \tag{6.4}$$

Now, according to the principle of local state we can take equation (1.18) to be valid locally*)

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*) In part II we do not adopt an assumption (3.8).

$$\dot{u}(Y^{s\varepsilon}) = T(Y^{s\varepsilon})\dot{s} + \frac{1}{\varrho}\underset{\sim}{\sigma}^{\circ}\cdot\dot{\underset{\sim}{\varepsilon}}^{(e)} + \frac{1}{\varrho}\underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}} + \frac{1}{\varrho}\pi\dot{b} \qquad (6.5)$$

This equation is also called "Gibbs fundamental equation". All equations of state (1.19) (or any other presented in table 1) remain also valid but the consideration presented in sec. 1.6 have to be modified since now not every elementary subsystem is surrounded by an ideal system and there is some energy dissipated in the process of heat transport. We can assume, however, that the whole body is surrounded by an ideal system B . Arguments similar to that presented in sec. (1.6), but applied to rates give

$$\dot{S}^{(B)} = \int_{S} \frac{\underset{\sim}{q}^{o(B)}\cdot\underset{\sim}{n}}{T^{(B)}}\, ds \qquad (6.6)$$

$$\dot{S}^{(A)} + \dot{S}^{(B)} \geqslant 0 \qquad (6.7)$$

where $\underset{\sim}{n}$ is outward unit normal to S , $\underset{\sim}{q}^{o(B)}$ heat flux delivered to B through the surface S ($\underset{\sim}{q}^{o(B)}\cdot\underset{\sim}{n} > 0$ when heat leaves the body A). Now, assuming that

$$\underset{\sim}{q}^{o(B)} = \underset{\sim}{q}^{(A)} \;,\quad T^{(A)} = T^{(B)} \qquad (6.8)$$

we get the Inequality of Clausius for continuous systems*) (superscript A is henceforth omitted)

$$\int_{D} \varrho\dot{s}\, dv + \int_{S} \frac{\underset{\sim}{q}\cdot\underset{\sim}{n}}{T}\, dS \geqslant 0 \qquad (6.9)$$

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*) Note, however that equations (6.7) is assumed directly in form of rates. If one does not adopt the postulat of local state the proper, generalization of (1.30) should be expressed in the form of time intergrals between two equilibrium states.

Here we have utilized the fact that entropy is additive and introduced the specific entropy s per unit mass. The inequality (6.9) can be equivalently expressed in the following form

$$\int_D \rho \dot{s}\, dv = -\int_S \frac{\underset{\sim}{q}\cdot \underset{\sim}{n}}{T}\, dS + \int_D \rho \underset{(i)}{\dot{s}}\, dv \tag{6.10}$$

$$\int_D \rho \underset{(i)}{\dot{s}}\, dv \geqslant 0 \tag{6.11}$$

Equation (6.10) represents the global form of the balance equation for the entropy whereas $\underset{(i)}{\dot{s}}$ is the rate of entropy production per unit mass. From (6.10) we obtain the following local expressions*)

$$\rho\, \dot{s} = -\,\mathrm{div}\left(\frac{\underset{\sim}{q}}{T}\right) + \rho \underset{(i)}{\dot{s}} \tag{6.12}$$

$$\underset{(i)}{\dot{s}} \geqslant 0 \tag{6.13}$$

By eliminating $\dot{u}$ and $\dot{s}$ between (6.2), (6.5) and (6.12) we find that

$$T\dot{s} = -\frac{1}{\rho}\,\mathrm{div}\,\underset{\sim}{q} + \underset{(D)}{\dot{w}} \tag{6.14}$$

$$\rho T \underset{(i)}{\dot{s}} = \rho \underset{(D)}{\dot{w}} - \frac{1}{T}\underset{\sim}{q}\cdot(\mathrm{grad}\, T) =$$

$$= \underset{\sim}{\sigma}^{d}\cdot \dot{\underset{\sim}{\varepsilon}}^{(e)} + \underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{\varepsilon}}^{(i)} - \underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}} - \pi \dot{b} - \frac{1}{T}\underset{\sim}{q}\cdot(\mathrm{grad}\, T) \geqslant 0 \tag{6.15}$$

where $\underset{(D)}{\dot{w}}$ is the rate of energy dissipation defined by (1.34). It is seen that equation (1.33) is the same as (6.14) (see 6.4). However the expression for entropy

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*) In fact (6.12) can be modified - see I. Müller [23].

production (6.15) contains a new term $-\underset{\sim}{q}\cdot(\mathrm{grad}\,T)/T$ (cf. eq. 1.35). This term represents energy loss dissipated due to transport of heat. We note that now a new quantity enters the theory: grad T , and we need an additional phenomenological equation which would relate "rate" $\underset{\sim}{q}$ with grad T and other dissipative forces. There is, however, no experimental evidence of thermodynamic coupling between the internal energy dissipation $\underset{(D)}{\dot{w}}$ and dissipation due to heat transport. Therefore we assume that $\underset{\sim}{q}$ depends on state and grad T alone

$$\mathrm{grad}\ T = \underset{\sim}{\psi_4}\,(\underset{\sim}{q}, Y_b^T) \tag{6.16}$$

Accounting for (3.1)-(3.4) and (6.16) we can deduce that if the inequality (6.15) is to be satisfied for arbitrary $\dot{\underset{\sim}{\varepsilon}}^{(e)}, \dot{\underset{\sim}{\varepsilon}}^{(i)}, \dot{\underset{\sim}{\beta}}, \dot{b}$ and $\underset{\sim}{q}$ then the following two inequalities must be satisfied

$$\underset{(D)}{\dot{w}} \geqslant 0 \quad , \quad \underset{\sim}{q}\cdot\underset{\sim}{\psi_4} \leq 0 \tag{6.17}$$

A particular form of the equation (6.16) is the well known Fouries Law for heat conduction

$$\underset{\sim}{q} = -\underset{\sim}{k}\ \mathrm{grad}\,T \ ; \quad k_{ij}\,T_{,i}\,T_{,j} > 0 \text{ for } T_{,i} = 0 \tag{6.18}$$

III EQUILIBRIUM PROPERTIES OF THERMOPLASTIC BEHAVIOUR

7. Reference State - Unloaded States

In this part we shall adopt the assumption (3.8). Morever we assume that the free energy φ has the following form

$$\varphi(Y^{T\varepsilon}) = \underset{(E)}{\varphi}(Y^T_\varepsilon) + \underset{(H)}{\varphi}(Y^T_b) \tag{7.1}$$

where (cf. 1.20)

$$Y^{T\varepsilon} = \{T, \underset{\sim}{\varepsilon}^{(e)}, \underset{\sim}{\beta}, b\}; \quad Y^T_\varepsilon = \{T, \underset{\sim}{\varepsilon}^{(e)}\}; \quad Y^T_b = \{T, \underset{\sim}{\beta}, b\} \tag{7.2}$$

It will become clear in sec. 8 that assumption (7.1) is equivalent to the neglection of the influence of parameters b and $\underset{\sim}{\beta}$ on elastic constants and thermal expansion coefficients.

From (7.1) it follows that the Gibbs and entropy functions can also be presented in the form of a sum

$$g(Y^{T\sigma}) = \underset{(E)}{g}(Y^T_\sigma) + \underset{(H)}{g}(Y^T_b) \tag{7.3}$$

$$s(Y^{T\varepsilon}) = \underset{(E)}{s}(Y^T_\varepsilon) + \underset{(H)}{s}(Y^T_b); \quad s(Y^{T\sigma}) = \underset{(E)}{s}(Y^T_\sigma) + \underset{(H)}{s}(Y^T_b) \tag{7.4}$$

where

$$Y^{T\sigma} = \{T, \underset{\sim}{\sigma}, \underset{\sim}{\beta}, b\}; \quad Y^T_\sigma = \{T, \underset{\sim}{\sigma}\} \tag{7.5}$$

$$\underset{(H)}{g}(Y^T_b) = \underset{(H)}{\varphi}(Y^T_b); \quad \underset{(E)}{g}(Y^T_\sigma) = \left[\underset{(E)}{\varphi} - \frac{1}{\varrho}\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}^{(e)}\right]_{\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{\varepsilon}^{(e)}(Y^T_\sigma)} \tag{7.6}$$

$$\underset{(E)}{s}(Y^T_\varepsilon) = -\frac{\partial \underset{(E)}{\varphi}(Y^T_\varepsilon)}{\partial T}; \quad \underset{(E)}{s}(Y^T_\sigma) = -\frac{\partial \underset{(E)}{g}(Y^T_\sigma)}{\partial T} \tag{7.7}$$

$$\underset{(H)}{s}(Y^T_b) = -\frac{\partial \underset{(H)}{\varphi}(Y^T_b)}{\partial T} \tag{7.8}$$

The function $\underset{(H)}{s}$ represents the change of entropy due to hardening associated with plastic deformation. Assumption (7.1) makes it possible to present also the internal energy function in the form of a sum provided that u is expressed in terms of*) $Y^{T\varepsilon}$ or $Y^{T\sigma}$, e. g.,

$$u(Y^{T\varepsilon}) = \underset{(E)}{u}(Y^T_\varepsilon) + \underset{(H)}{u}(Y^T_b) \tag{7.9}$$

where

$$\begin{aligned} \underset{(E)}{u}(Y^T_\varepsilon) &= \underset{(E)}{\varphi}(Y^T_\varepsilon) + T\underset{(E)}{s}(Y^T_\varepsilon) \\ \underset{(H)}{u}(Y^T_b) &= \underset{(H)}{\varphi}(Y^T_b) + T\underset{(H)}{s}(Y^T_b) \end{aligned} \tag{7.10}$$

The function $\underset{(H)}{u}(Y^T_b)$ represents <u>stored energy</u> due to isothermal cold work [25], [18] and will be briefly discussed in sec. 9.

Among all possible constrained states of a uniform material system we distinguish the <u>unloaded (natural) states</u> which are characterized by

$$T = T_o\;; \quad \underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{0}\;; \quad \underset{\sim}{\sigma} = \underset{\sim}{0} \tag{7.11}$$

Here T_o is a certain reference temperature. The unloaded state for which

$$\underset{\sim}{\beta} = \underset{\sim}{0}\;; \quad b = 0 \tag{7.12}$$

we shall call <u>reference state</u>. From (7.11) it follows that

$$\left.\frac{\partial \underset{(E)}{\varphi}}{\partial \underset{\sim}{\varepsilon}^{(e)}}\right|_{\substack{\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{0} \\ T = T_o}} = \underset{\sim}{0}\;; \qquad \left.\frac{\partial \underset{(E)}{g}}{\partial \underset{\sim}{\sigma}}\right|_{\substack{\underset{\sim}{\sigma} = \underset{\sim}{0} \\ T = T_o}} = \underset{\sim}{0} \tag{7.13}$$

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*) The function u expressed in terms of those variables is not a thermodynamical potential.

Without any physical restrictions we can also assume that

$$\underset{(H)}{\varphi} = 0 \quad \text{for} \quad \underset{\sim}{\beta} = \underset{\sim}{0}, \; b = 0 \tag{7.14}$$

Hence,

$$-\underset{(H)}{s} = \frac{\partial \underset{(H)}{\varphi}}{\partial T} = 0 \quad \text{for} \quad \underset{\sim}{\beta} = \underset{\sim}{0}, \; b = 0 \tag{7.15}$$

Consider an isothermal process in a uniform system $(\dot{T} = 0, \; T_{,i} = 0)$. In the course of such process the rate of function $\underset{(H)}{\varphi}$ is exactly equal to $\underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}}/\rho + \pi\dot{b}/\rho$ (cf.(1.27), (1.28) and (7.1)), and under assumption (3.8) it follows from (2.2) that [6]

$$\underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{\varepsilon}}^{(i)} \geqslant \rho\,\dot{\underset{(H)}{\varphi}} \tag{7.16}$$

It is seen that in the absence of stresses the potential $\underset{(H)}{\varphi}$ can only decrease. This fact is independent of particular plastic properties of the material under consideration, and it indicates the "direction" of possible spontaneous, isothermal changes of internal state in absence of stresses.

8. Thermostatic Coupled Effects in Thermoplasticty

8.1. Basic Definitions

Let us introduce the following definitions of the physical quantities that characterize the thermo-mechanical properties of a material system:

(i) The isothermal elastic stiffness $\underset{\sim}{L}$ and compliances $\underset{\sim}{M}$

$$\underset{\sim}{L}(Y^{T\varepsilon}) = \frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} = \rho\,\frac{\partial^2 \varphi(Y^{T\varepsilon})}{\partial^2 \underset{\sim}{\varepsilon}^{(e)}} \tag{8.1}$$

$$L_{ijmn} = \rho\,\frac{\partial^2 \varphi}{\partial \varepsilon^{(e)}_{ij}\,\partial \varepsilon^{(e)}_{mn}}$$

$$\underset{\sim}{M}\,(Y^{T\sigma}) = \frac{\partial \underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma})}{\partial \underset{\sim}{\sigma}} = -\,\rho\,\frac{\partial^2 g(Y^{T\sigma})}{\partial^2 \underset{\sim}{\sigma}} \qquad (8.2)$$

(ii) Specific heat c_ε at constant $\underset{\sim}{\varepsilon}^{(e)}$ and specific heat c_σ at constant stress

$$c_\varepsilon\,(Y^{T\varepsilon}) = T\,\frac{\partial s(Y^{T\varepsilon})}{\partial T} = \frac{\partial u(Y^{T\varepsilon})}{\partial T} = -\,T\,\frac{\partial^2 \varphi(Y^{T\varepsilon})}{\partial T^2} \qquad (8.3)$$

$$c_\sigma\,(Y^{T\sigma}) = T\,\frac{\partial s(Y^{T\sigma})}{\partial T} = -\,T\,\frac{\partial^2 g(Y^{T\sigma})}{\partial T^2} \qquad (8.4)$$

(iii) Thermal Expansion tensor $\underset{\sim}{\alpha}$

$$\underset{\sim}{\alpha}\,(Y^{T\sigma}) = \frac{\partial \underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma})}{\partial T} = -\,\rho\,\frac{\partial^2 g(Y^{T\sigma})}{\partial \underset{\sim}{\sigma}\,\partial T} \qquad (8.5)$$

Consider the identity

$$\frac{1}{\rho}\,\underset{\sim}{\sigma} = \left.\frac{\partial \varphi(Y^{T\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}}\right|_{\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma})} \qquad (8.6)$$

By calculating the partial derivatives from both sides of the above equation with respect to $\underset{\sim}{\sigma}$ we get

$$\underset{\sim}{M} = \underset{\sim}{L}^{-1} \quad \text{or} \quad 2\,L_{ijrs}\,M_{rsmn} = \delta_{im}\,\delta_{jn} + \delta_{jm}\,\delta_{in} \qquad (8.7)$$

Now, calculation of the partial derivative of both sides of eq. (8.6) with respect to T gives

$$\underset{\sim}{Q} = \frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial T} + \frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\varepsilon}^{(e)}} \frac{\partial \underset{\sim}{\varepsilon}^{(e)}}{\partial T}$$

Hence

$$\frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial T} = - \underset{\sim}{L}(Y^{T\varepsilon}) \underset{\sim}{\alpha} = - \varrho \frac{\partial s(Y^{T\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} \tag{8.8}$$

Finally, let us derive the relation between c_σ and c_ε by differentiating the equality

$$s(Y^{T\varepsilon})\Big|_{\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma})} = s(Y^{T\sigma}) .$$

with respect to temperature

$$\frac{\partial s(Y^{T\varepsilon})}{\partial T} + \frac{\partial s(Y^{T\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} \cdot \frac{\partial \underset{\sim}{\varepsilon}^{(e)}}{\partial T} = \frac{\partial s(Y^{T\sigma})}{\partial T}$$

Accounting for (8.3)-(8.5) and (8.8) we get the following relation

$$c_\sigma(Y^{T\sigma}) = \left[c_\varepsilon + \frac{T}{\varrho} \underset{\sim}{\alpha} \cdot (\underset{\sim}{L} \underset{\sim}{\alpha}) \right]_{\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma})} \tag{8.9}$$

Note that to justify identities (8.7)-(8.9) we have not used assumption (7.1). Note also that if (7.1) holds then $\underset{\sim}{L}$, $\underset{\sim}{M}$ and $\underset{\sim}{\alpha}$ are independent of $\underset{\sim}{\beta}$ and b . However c_σ and c_ε can still be influenced by the hardening.

Sometimes one uses so called adiabatic stiffness $\overset{(A)}{\underset{\sim}{L}}$ defined by

$$\overset{(A)}{\underset{\sim}{L}}(Y^{s\varepsilon}) = \frac{\partial \underset{\sim}{\sigma}(Y^{s\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} = \rho \, \frac{\partial^2 u(Y^{s\varepsilon})}{\partial^2 \underset{\sim}{\varepsilon}^{(e)}} \tag{8.10}$$

Let us find the relation between $\overset{(A)}{\underset{\sim}{L}}$ and $\underset{\sim}{L}$.

$$\frac{\partial \underset{\sim}{\sigma}(Y^{s\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} = \frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} + \frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial T} \otimes \frac{\partial T(Y^{s\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} \tag{8.11}$$

But

$$\frac{\partial T(Y^{s\varepsilon})}{\partial \underset{\sim}{\varepsilon}^{(e)}} = \frac{1}{\rho}\,\frac{\partial \underset{\sim}{\sigma}(Y^{s\varepsilon})}{\partial s} = \frac{1}{\rho}\,\frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial T}\,\frac{\partial T(Y^{s\varepsilon})}{\partial s} = \frac{1}{\rho}\,\frac{\partial \underset{\sim}{\sigma}(Y^{T\varepsilon})}{\partial T}\,\frac{T}{\frac{\partial s(Y^{T\varepsilon})}{\partial T}\,T} \tag{8.12}$$

Accounting for (8.8), (8.3) and substituting (8.12) into (8.11) we finally get

$$\overset{(A)}{\underset{\sim}{L}}(Y^{s\varepsilon}) = \underset{\sim}{L}(Y^{T\varepsilon}) + \frac{1}{\rho}\,\frac{T}{c_\varepsilon}\,(\underset{\sim}{L}\underset{\sim}{\alpha}) \otimes (\underset{\sim}{L}\underset{\sim}{\alpha}) \tag{8.13}$$

One usually deals with so called intrinsic stable materials with respect to thermoelastic properties. For such materials the matrix of second order derivatives of u with respect to s and $\underset{\sim}{\varepsilon}^{(e)}$ is positive definite*). The conditions of positive-definitness imply

$$c_\varepsilon > 0 \qquad d\underset{\sim}{\varepsilon}^{(e)} \cdot \underset{\sim}{L}\, d\underset{\sim}{\varepsilon}^{(e)} > 0 \text{ for every } d\underset{\sim}{\varepsilon}^{(e)} \neq \underset{\sim}{0}$$

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*) The physical content of this stability criterion is known as La Chatelier's principle [2]. Recently more common is to require the function u to be convex with respect to s and $\underset{\sim}{\varepsilon}^{(e)}$.

Thus the tensor $\underset{\sim}{L}$ is positive-definite. Note that from (8.11) it follows that $\underset{\sim}{L}^{(A)}$ is also positive-definite, whereas (8.9) indicates that $c_\sigma > c_\varepsilon$.

8.2. Coupled Effects

To discuss some coupled effects let us calculate the total differentials of dependent state variables assuming in sequence: the set $Y^{T\varepsilon}$ and $Y^{T\sigma}$ as a set of independent state variables. Accounting for assumption (7.1), definitions (8.1)-(8.5) and (8.8) we find

$$T ds(Y^{T\varepsilon}) = \underbrace{c_\varepsilon dT}_{\substack{\text{heat}\\ \text{capaci-}\\ \text{ty}}} + \underbrace{\frac{T}{\varrho} d\underset{\sim}{\varepsilon}^{(e)}\cdot(\underset{\sim}{L}\underset{\sim}{\alpha})}_{\substack{\text{heat of el.}\\ \text{deformation}}} + \underbrace{T\Big[\frac{\partial s_{(H)}}{\partial \underset{\sim}{\beta}}\cdot d\underset{\sim}{\beta} + \frac{\partial s_{(H)}}{\partial b}\, db\Big]}_{\substack{\text{heat of internal}\\ \text{structural changes}\\ T\,d s_{(H)}}} \tag{8.12}$$

$$T ds(Y^{T\sigma}) = \overbrace{c_\sigma dT} + \underbrace{\frac{T}{\varrho}\underset{\sim}{\alpha}\cdot d\underset{\sim}{\sigma}}_{\substack{\text{thermostatic}\\ \text{piezocaloric}\\ \text{effect}}} + \overbrace{T\Big[\frac{\partial s_{(H)}}{\partial \underset{\sim}{\beta}}\cdot d\underset{\sim}{\beta} + \frac{\partial s_{(H)}}{\partial b}\, db\Big]} \tag{8.13}$$

$$d\underset{\sim}{\sigma}(Y^{T\varepsilon}) = \underbrace{\underset{\sim}{L}\, d\underset{\sim}{\varepsilon}^{(e)}}_{\text{elasticity}} - \underbrace{\underset{\sim}{L}\underset{\sim}{\alpha}\, dT}_{\text{elastic thermal stresses}} \tag{8.14}$$

$$d\underset{\sim}{\varepsilon}^{(e)}(Y^{T\sigma}) = \overbrace{\underset{\sim}{M}\, d\underset{\sim}{\sigma}} + \underbrace{\underset{\sim}{\alpha}\, dT}_{\text{thermal expansion}} \tag{8.15}$$

$$d\underset{\sim}{\Pi} = \underbrace{\frac{\partial\underset{\sim}{\Pi}}{\partial\underset{\sim}{\beta}}\cdot d\underset{\sim}{\beta} + \frac{\partial\underset{\sim}{\Pi}}{\partial b}\,db}_{\text{internal structural changes}} + \underbrace{\frac{\partial\underset{\sim}{\Pi}}{\partial T}\,dT}_{\text{internal thermal forces}} \qquad (8.16)$$

$$d\pi = \overbrace{\frac{\partial\pi}{\partial\underset{\sim}{\beta}}\cdot d\underset{\sim}{\beta} + \frac{\partial\pi}{\partial b}\,db} + \overbrace{\frac{\partial\pi}{\partial T}\,dT} \qquad (8.17)$$

The interpretation of various terms occuring in eqs. (8.12)-(8.17) follows directly from the physical sense of corresponding partial derivatives. For example, $T\,\partial s(Y^{T\sigma})/\partial\underset{\sim}{\sigma}$ represents the heat produced in an experiment in the course of which $\underset{\sim}{\beta}$, b and T are kept constant but stress varies. This effect is called "thermostatic piezocaloric effect". The consequence of the assumption (7.1) is, among other, prediction that the internal changes have no influence on the above mentioned heat.

The names associated with term $\bar{d}s_{(H)}$ and both terms in eq. (8.17) are rather general. One can not give more accurate interpretation until the concrete physical meaning is attached to the internal parameters $\underset{\sim}{\beta}$ and b . Nevertheless we have introduced such general names in order to underline the fact that new thermostatic effects may occur when the body is plastically strained.

The possible couplings are illustrated*) by the diagram in fig. 3. In three outer corners are temperature T , stress $\underset{\sim}{\sigma}$ and internal reversible forces $\underset{\sim}{\Pi}$ and $\underset{\sim}{\pi}$. In the three corresponding inner corners appear entropy s,

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*) This is a development of a diagram given by Nye [26].

elastic strain $\underset{\sim}{\varepsilon}^{(e)}$ and internal coordinates $\underset{\sim}{\beta}$, b . All these quantities are "direct" results of changes of T , $\underset{\sim}{\sigma}$, $\underset{\sim}{\Pi}$ and π . The double lines represents the "principal effects". The diagram also illustrates what is called the "coupled effects". These are represented by lines joining pairs of points which are not both at the same corner. The lines connecting T , s , $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}^{(e)}$ denote the coupled effect of thermoelastic behaviour, whereas the dotted lines indicate couplings which are excluded from the consideration due to the assumption (7.1).

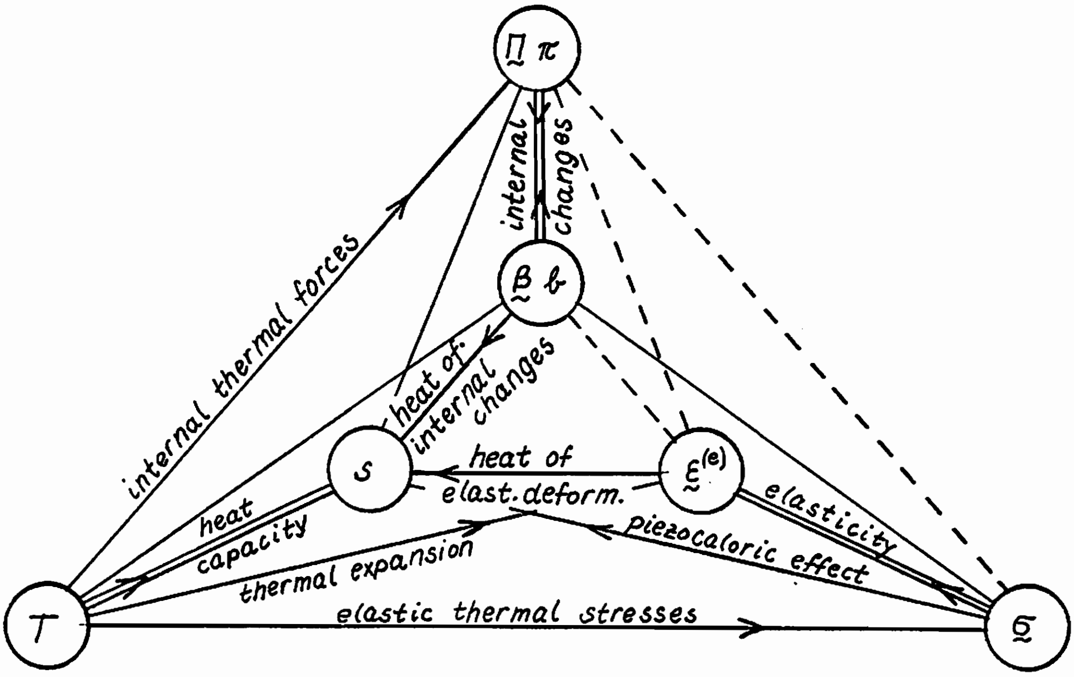

Fig. 3. The diagram of some coupled effects in thermoplasticty.

9. On the Stored Energy of Isothermal Plastic Deformations

The metal is said to be cold worked if it is plastically deformed at the temperature less than (approximately) half of the melting temperature. The most of mechanical work expanded in the isothermal process of cold work is converted into heat, but the remainder is stored in the metal changing its internal energy. The experimental investigations show that the part of mechanical work that remains in a metal may vary from 2 % to 15 %.

Consider an isothermal (T = const.) cyclic process in $\underset{\sim}{\sigma}$ space which begins at some state 1 and ends at another state 2. From (7.9) we get

$$\overset{(2)}{u}-\overset{(1)}{u}=\int_1^2 d\underset{(E)}{u}+\int_1^2 d\underset{(H)}{u}=\underset{(H)}{u}\left(\overset{(2)}{Y}{}^T_b\right)-\underset{(H)}{u}\left(\overset{(1)}{Y}{}^T_b\right) \qquad (9.1)$$

where

$$\overset{(2)}{Y}{}^T_b=\left\{T,\underset{\sim}{\beta}_2,b_2\right\};\qquad \overset{(1)}{Y}{}^T_b=\left\{T,\underset{\sim}{\beta}_1,b_1\right\}$$

On the other hand from the first law (1.4) we obtain (cf. 1.6)

$$\overset{(2)}{u}-\overset{(1)}{u}=\int_1^2 dq+\int_1^2\frac{1}{\varrho}\underset{\sim}{\sigma}(Y^T_\varepsilon)\cdot d\underset{\sim}{\varepsilon}^{(e)}+\int_1^2\frac{1}{\varrho}\underset{\sim}{\sigma}\cdot d\underset{\sim}{\varepsilon}^{(i)}=$$

$$=Q(1,2)+\underset{(i)}{W}(1,2) \qquad (9.2)$$

The second integral in (9.2) vanishes since T = const., $\underset{\sim}{\sigma}$ is a state function and depends only on T and $\underset{\sim}{\varepsilon}^{(e)}$ (cf. 7.1), and $\underset{\sim}{\varepsilon}^{(e)}_2=\underset{\sim}{\varepsilon}^{(e)}_1$. Whence

$$\underset{(H)}{u}\left(\overset{(2)}{Y}{}^T_b\right)-\underset{(H)}{u}\left(\overset{(1)}{Y}{}^T_b\right)=Q(1,2)+\underset{(i)}{W}(1,2) \qquad (9.3)$$

If the state 1 is choosen to be such that $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, $b_1 = 0$ then from (7.10), (7.14) and (7.15) it follows that $\underset{(H)}{u}(\overset{(1)}{y}{}^T_b) = 0$. It is seen that $\underset{(H)}{u}(y^T_b)$ represents stored energy due to isothermal cold work from the state at which $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, $b_1 = 0$. It is equal to the difference between work expanded in cold work and heat $-Q\,(1,2)$ which is absorbed by surrounding. This remark is illustrated in fig. 4 on the background of simple tension curve. Note that to derive (9.3) we have not used any "constitutive equations",

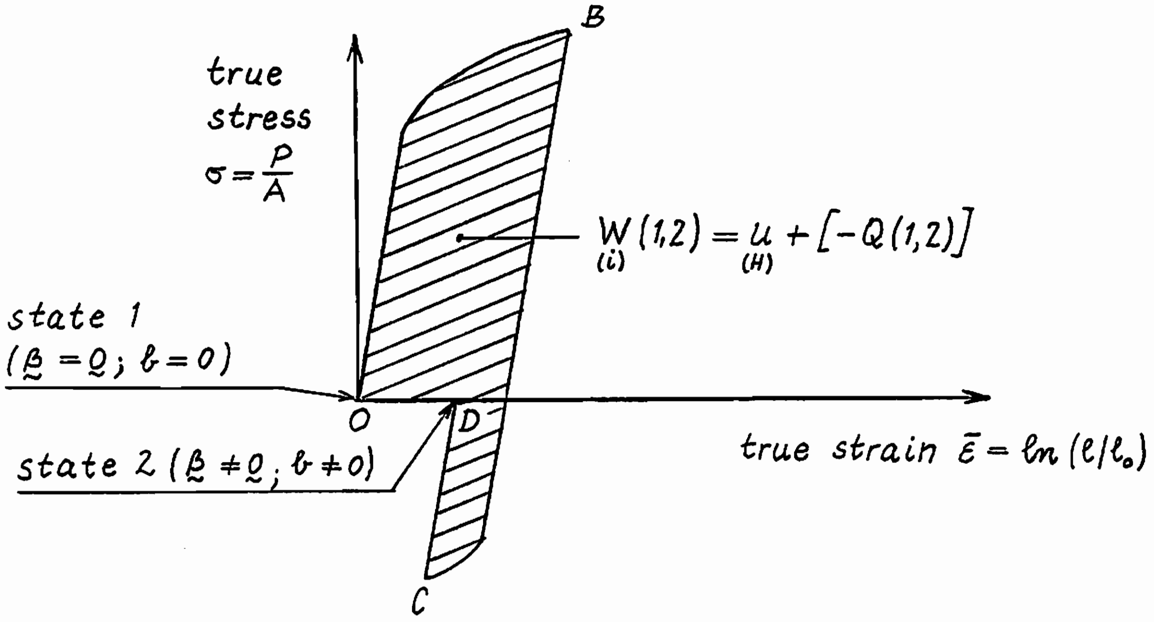

Fig. 4. The inelastic work (area under the curve $OBCD$) is not equal to the heat absorbed by a surrounding. A - actual area of cross-section, ℓ - actual length, ℓ_0 - initial length, P - force.

i. e. (9.3) is valid independently of particular properties of a metal under consideration, provided (7.1) holds. The heat $-Q\,(1,2)$ which takes part in such kind of experiments, however, is not generally equal to energy dissipation. Indeed, from (2.2), (1.28), (3.8) and (7.1) it follows that

$$\underset{(D)}{W}(1,2) = \underset{(i)}{W}(1,2) - \underset{(H)}{\varphi}(1,2) \tag{9.4}$$

where

$$\underset{(D)}{W}(1,2) = \int_1^2 d\underset{(D)}{w}\ ; \quad \underset{(H)}{\varphi}(1,2) = \underset{(H)}{\varphi}\left(\overset{(2)}{\gamma}{}^{T}_{\ell}\right) - \underset{(H)}{\varphi}\left(\overset{(1)}{\gamma}{}^{T}_{\ell}\right)$$

or through (9.4), (7.10) and (9.3)

$$\underset{(D)}{W}(1,2) = -Q(1,2) - T\left[\underset{(H)}{s}\left(\overset{(2)}{\gamma}{}^{T}_{\ell}\right) - \underset{(H)}{s}\left(\overset{(1)}{\gamma}{}^{T}_{\ell}\right)\right] \qquad (9.5)$$

It is seen that the heat absorbed by the surrounding $-Q\,(1,2)$ when reduced by the heat of internal structural changes (cf. 8.12) is equal to the total energy dissipation in a cyclic isothermal process in $\underset{\sim}{g}$ -space. While inelastic deformations certainly change the entropy of a metal the heat of internal changes is believed to be small compared to the increase of internal energy. The term $T\underset{(H)}{s}$ in (7.10) is, therefore, frequently neglected, so that the free energy increase is equated directly to stored energy.

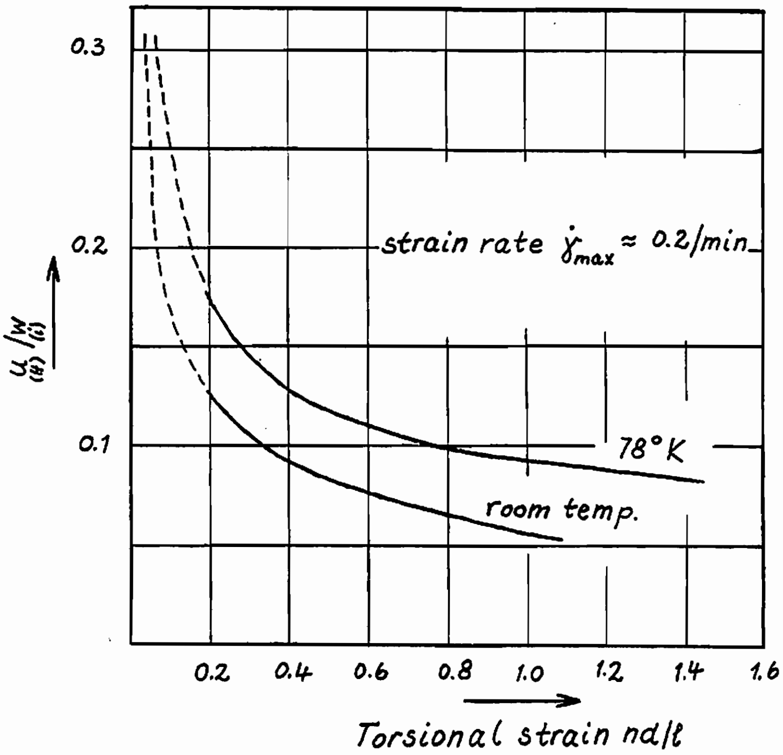

Fig. 5. The ratio $\underset{(H)}{u}/\underset{(i)}{W}$ as a function of strain in an 82.6Au-17.4Ag alloy deformed by torsion at 78°K and room temperature. Appleton and Bever.

In fig. 5-7 the results of experiments made by Appleton and Bever, and Degtiarev are presented (see [25] pg. 54-55 and 62). For torsion strain is expressed as nd/ℓ where n is the number of turns, d diameter of the specimen and ℓ its length. These figures contain some informations concerning the character of the changes of $u_{(H)}$ with temperature and primary hardening. However one should rather be careful when using such data to determine the thermostatic properties of a metal. There are a number of techniques of measurements of stored energy and there are some uncertainities wheather particular technique concerns stored energy due to isothermal plastic deformation

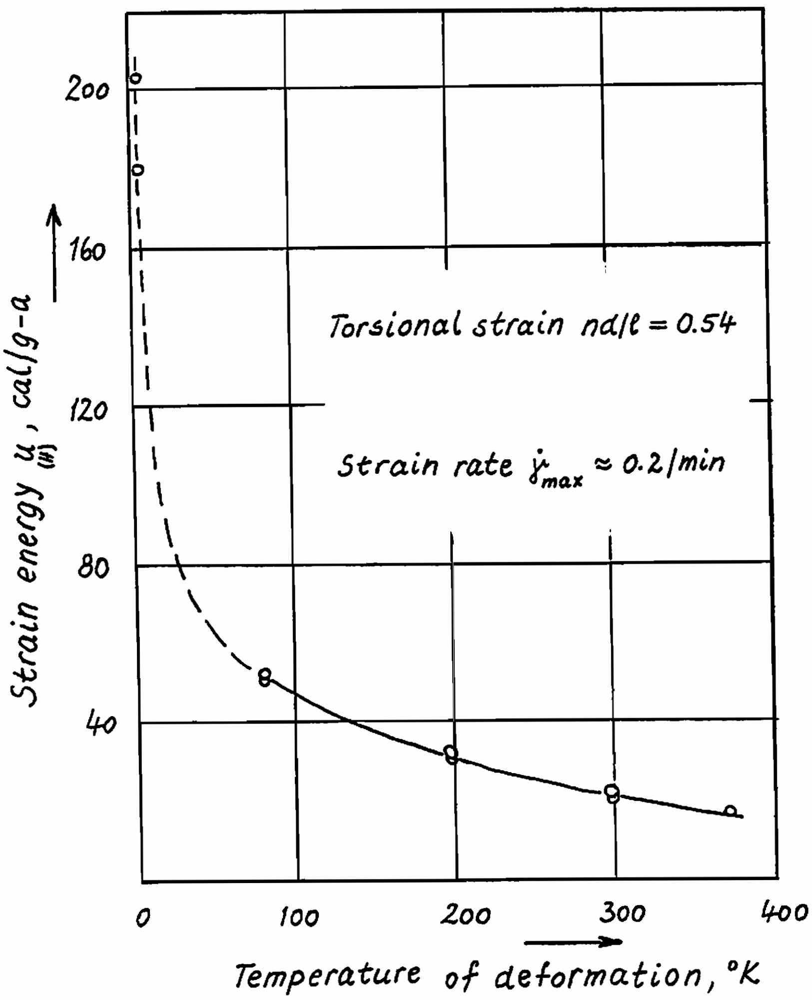

Fig. 6. The stored energy as a function of temperature of deformation in an 82.6Au-17.4Ag alloy deformed by torsion. Appleton and Bever.

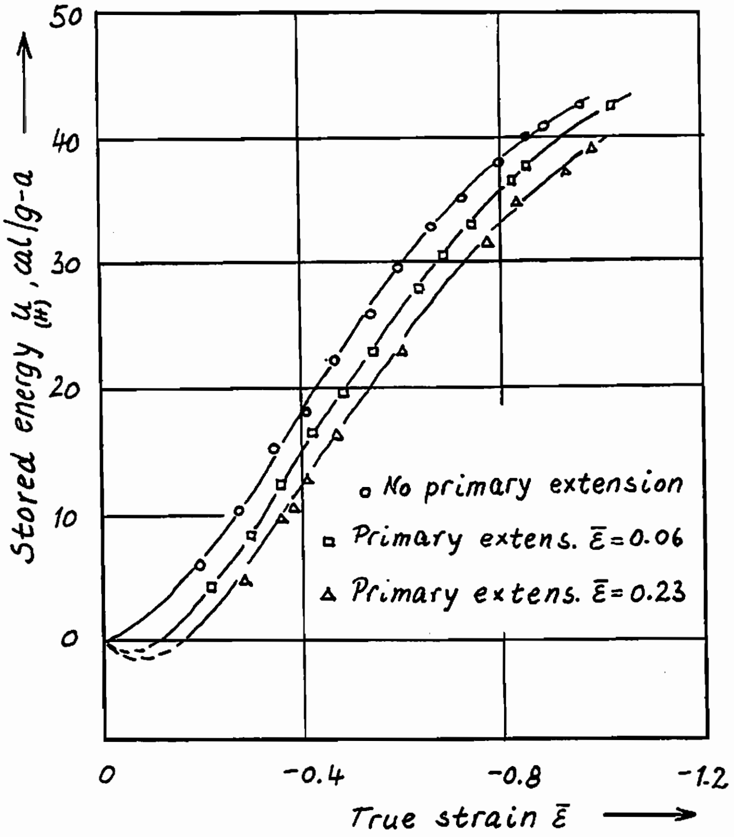

Fig. 7. Energy stored in secondary compression of copper after various amounts of primary extension. Degtiarev, Polakowski.

or any other conditions. In the latter case the stored energy is generally not a state function such as $\underset{(H)}{u}$ (see [25]).

In the so called "recovery" stage of annealing, the physical properties (e. g. yield stress) that suffered changes as a result of plastic deformations tend to recover their original values. This is done on expense of stored energy (cf. [27]).

10. Free Energy of a Material with Constant $\underset{\sim}{M}$ and $\underset{\sim}{\alpha}$

Let us assume that $\underset{\sim}{M}$ and $\underset{\sim}{\alpha}$ are known and are independent of $\underset{\sim}{\sigma}$ and T . As a consequence the specific heats c_ε and c_σ are merely functions of T , $\underset{\sim}{\beta}$ and b since from (7.1), (7.7), (8.4) and (8.5) it follows that $\partial \underset{(E)}{s} / \partial \underset{\sim}{\sigma}$ = const. Imagine that the measurements of $\underset{(H)}{u}$ and c_ε and c_σ have been done so that these quantities are known functions of T , $\underset{\sim}{\beta}$ and b . Under the adopted assumptions concerning $\underset{\sim}{Y}$ and $\underset{\sim}{\alpha}$ one can express $\underset{(H)}{u}$, c_ε and c_σ in the form

$$c_\varepsilon(Y^T_b) = c^{\circ}_\varepsilon(T) + \Delta c(Y^T_b)$$

$$c_\sigma(Y^T_b) = c'_\sigma(T) + \Delta c(Y^T_b) \tag{10.1}$$

$$\underset{(H)}{u}(Y^T_b) = \underset{(H)}{\bar{u}}(\underset{\sim}{\beta}, b) + \Delta \underset{(H)}{u}(Y^T_b)$$

where

$$\Delta c = \underset{(H)}{\bar{u}} = \Delta \underset{(H)}{u} = 0 \quad \text{for} \quad \underset{\sim}{\beta} = \underset{\sim}{0},\ b = 0$$

$$\Delta \underset{(H)}{u} = 0 \quad \text{for} \quad T = T_0 \tag{10.2}$$

Here $c^{\circ}_\varepsilon(T)$ and $c'_\sigma(T)$ are the specific heats measured from the reference state ($\underset{\sim}{\beta} = \underset{\sim}{0}$, $b = 0$, $T = T_0$; unstrained plastically test speciments). Δc represents the difference between c_ε (or c_σ) measured from the arbitrary unloaded state and c°_ε (or c'_σ) . The equality

$$c_\varepsilon - c^{\circ}_\varepsilon = c_\sigma - c'_\sigma$$

follows from the identity (8.9). From this identity it follows also that

$$c'_\sigma = c^{\circ}_\varepsilon(T) + \frac{T}{\rho}\underset{\sim}{\alpha} \cdot \underset{\sim}{L}\,\underset{\sim}{\alpha} \tag{10.3}$$

Function $\underset{(H)}{\bar{u}}(\underset{\sim}{\beta}, b)$ represents the stored energy due to isothermal cold-work at reference temperature T_0 whereas $\Delta \underset{(H)}{u}(Y^T_b)$ is the difference between the stored energy measured at $T \neq T_0$ and $\underset{(H)}{\bar{u}}$. Note also that from the second equality (8.3) it follows that

$$\frac{\partial}{\partial T} \Delta \underset{(H)}{u} = \Delta c\left(Y_{b}^{T}\right) \tag{10.4}$$

i. e., $\Delta \underset{(H)}{u}$ and Δc are not independent.

Now, by integrating the equations (8.1) and (8.5) we find the following expressions for the thermodynamics potentials $\underset{(E)}{\varphi}$, $\underset{(H)}{\varphi}$, $\underset{(E)}{g}$ (cf. (7.1), (7.3) and (7.6))

$$\begin{aligned} \underset{(E)}{\varphi}\left(Y_{\varepsilon}^{T}\right) = {} & \frac{1}{2\rho} \underset{\sim}{\varepsilon}^{(e)} \cdot \underset{\approx}{L} \underset{\sim}{\varepsilon}^{(e)} - \frac{1}{\rho} \underset{\approx}{\alpha} \cdot \underset{\approx}{L} \underset{\sim}{\varepsilon}^{(e)} (T-T_0) + \int_{T_0}^{T} c_{\varepsilon}'(T_1)\, dT_1 - \\ & - T \int_{T_0}^{T} \frac{1}{T_1} c_{\varepsilon}'(T_1)\, dT_1 - s_0 (T-T_0) + \varphi_0 \end{aligned} \tag{10.5}$$

$$\begin{aligned} \underset{(E)}{g}\left(Y_{\sigma}^{T}\right) = {} & - \frac{1}{2\rho} \underset{\sim}{\sigma} \cdot \underset{\approx}{M} \underset{\sim}{\sigma} - \frac{1}{\rho} \underset{\approx}{\alpha} \cdot \underset{\sim}{\sigma} (T-T_0) + \int_{T_0}^{T} c_{\sigma}'(T_1)\, dT_1 - \\ & - T \int_{T_0}^{T} \frac{1}{T_1} c_{\sigma}'(T_1)\, dT_1 - s_0 (T-T_0) + \varphi_0 \end{aligned} \tag{10.6}$$

$$\begin{aligned} \underset{(H)}{\varphi}\left(Y_{b}^{T}\right) = {} & \int_{T_0}^{T} \Delta c\left(Y_{b}^{T_1}\right) dT_1 - T \int_{T_0}^{T} \frac{1}{T_1} \Delta c\left(Y_{b}^{T_1}\right) dT_1 - \\ & - \underset{(H)}{\bar{s}} (\underset{\sim}{\beta}, b) T + \underset{(H)}{\bar{u}} (\underset{\sim}{\beta}, b) \end{aligned} \tag{10.7}$$

where

$$\underset{H}{\bar{s}} = 0 \text{ for } \underset{\sim}{\beta} = \underset{\sim}{0}, \; b = 0$$

s_0 and φ_0 denote entropy and free energy, respectively, in the reference state.

For isotropic materials

$$\underset{\sim}{\alpha} = \alpha \underset{\sim}{1}$$

$$L_{ijmn} = \mu\left(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}\right) + \lambda\,\delta_{ij}\delta_{mn} \qquad (10.8)$$

$$M_{ijmn} = \frac{1}{4\mu}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) - \frac{\lambda}{2\mu(2\mu+3\lambda)}\,\delta_{ij}\delta_{mn}$$

where λ and μ are Lame's constant of ordinary infinitesimal elasticity. One can also simplify the expressions (10.5)-(10.7) expanding $\underset{(H)}{\varphi}$ and $\underset{(E)}{\varphi}$ into the Taylor series at point $\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{0}$, $T = T_0$, $\underset{\sim}{\beta} = \underset{\sim}{0}$, $b = 0$ and retaining only the terms of second order.

Calculating the entropy $\underset{(H)}{s}$ (cf. 7.15) we obtain

$$\underset{(H)}{s}\left(Y^{T}_{b}\right) = \underset{(H)}{\bar{s}}\left(\underset{\sim}{\beta}, b\right) + \Delta \underset{(H)}{s}\left(Y^{T}_{b}\right) \qquad (10.9)$$

where

$$\Delta \underset{(H)}{s}\left(Y^{T}_{b}\right) = \int_{T_0}^{T_1} \frac{1}{T_1}\,\Delta c\left(Y^{T_1}_{b}\right) dT_1 = \int_{T_0}^{T_1} \frac{1}{T_1}\frac{\partial}{\partial T_1}\,\Delta \underset{(H)}{u}\left(Y^{T_1}_{b}\right) dT_1 \qquad (10.10)$$

and $\underset{(H)}{\bar{s}}$ represents the difference between the entropy in the reference state and the entropy in an arbitrary unloaded state. This simple example illustrates the well known fact that the entropy $\underset{(H)}{\bar{s}}$ can not be determined neither through measurement of specific heats nor energy stored in isothermal process of plastic deformations. The experimental determination of the function $\underset{(H)}{\bar{s}}$ is one of the actual unsolved problems of the thermostatic of plasticity. It occurs as a consequence of our inability to contrive a real process which would be close to the assumed hypothetical reversible process at the very beginning of the development of the theory. Perhaps the partial solution may be found in reinterpretation of the so called "isothermal annealing" technique of measurement of stored energy?

Generally one expects help in this matter from statistical physics. Still another view is that these difficulties one will overcome by well guess of the form of the dissipation potential [28]. From the point of view of applied thermoplasticity, however, this problem is not very important since it is believed that the values of the term $\underset{(H)}{\bar{s}}\, T$ is rather less than that of $\underset{(H)}{\bar{u}}$ [27]. Since, on the other hand the plastic deformations affect specific heat negligibly, the assumption $\underset{(H)}{s} = 0$ is rather justified. Such simplification is frequently assumed in applications. It is equivalent with the assumption that the entropy of an elastic-plastic material is the same as the entropy of an elastic material.

11. Equations for Temperature in Thermoplasticity

The fundamental set of field equations in thermoplasticity can be expressed in terms of fields $T(\underset{\sim}{x},t)$ $\underset{\sim}{\varepsilon}(\underset{\sim}{x},t)$ (total strain) $\underset{\sim}{\sigma}(\underset{\sim}{x},t)$, $\underset{\sim}{\beta}(\underset{\sim}{x},t)$, $b(\underset{\sim}{x},t)$ and $\underset{\sim}{u}(\underset{\sim}{x},t)$ alone, if one assumes that one of the thermodynamic potentials (e. g. g), the function F_1 (occuring in (3.15)-(3.17)) and ψ_4 (cf. 6.16) are known. It consists of the 23 equations:

(i) 3 equilibrium equations (6.1)

(ii) 6 kinematical relations (6.3)

(iii) 7 equations $(3.15)_{2,3}$, $(3.16)_{2,3}$ for the internal coordinates $\underset{\sim}{\beta}$ and b . The $\underset{\sim}{\Pi}$ and π have to be eliminated from these equation by use of $(1.27)_3$ and $(1.28)_3$.

(iv) 6 strain-stress relations (cf. 8.15)

$$\dot{\underset{\sim}{\varepsilon}} = \underset{\sim}{M}\dot{\underset{\sim}{\sigma}} + \underset{\sim}{\alpha}\dot{T} + \dot{\underset{\sim}{\varepsilon}}^{(i)}$$

where $\dot{\underset{\sim}{\varepsilon}}^{(i)}$ is given by $(3.15)_1$ and $(3.16)_1$.

(v) One equation for temperature which various alternative forms we shall now derive.

Eliminating $T\dot{s}$ between (6.14) and (8.12) or (8.13) gives two alternative forms of the equation for temperature.

$$\rho c_{\varepsilon}\dot{T} = \rho \underset{(D)}{\dot{w}} \underbrace{- T\dot{\underset{\sim}{\varepsilon}}^{(e)}\cdot \underset{\sim}{L}\,\underset{\sim}{\alpha}}_{\substack{\text{due to heat of}\\ \text{elast. deformation}}} \underbrace{- \rho T\left[\frac{\partial \underset{(H)}{s}}{\partial \underset{\sim}{\beta}}\cdot \dot{\underset{\sim}{\beta}} + \frac{\partial \underset{(H)}{s}}{\partial b}\dot{b}\right]}_{\substack{\text{due to heat of}\\ \text{internal changes}}} - div\,\underset{\sim}{q} \qquad (11.1)$$

$$\rho c_{\sigma}\dot{T} = \rho \underset{(D)}{\dot{w}} \underbrace{- T\underset{\sim}{\alpha}\cdot\dot{\underset{\sim}{\sigma}}}_{\text{due to piezocaloric effect}} \overbrace{- \rho T\left[\frac{\partial \underset{(H)}{s}}{\partial \underset{\sim}{\beta}}\cdot \dot{\underset{\sim}{\beta}} + \frac{\partial \underset{(H)}{s}}{\partial b}\dot{b}\right]} - div\,\underset{\sim}{q} \qquad (11.2)$$

where $\underset{(D)}{\dot{w}}$ is energy dissipation given by (cf. 1.34 and 3.8)

$$\underset{(D)}{\dot{w}} = \frac{1}{\rho}\left(\underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{\varepsilon}}^{(i)} - \underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}} - \pi\dot{b}\right) \qquad (11.3)$$

Here q can be eliminated by use of e. g. Fourier's Law (6.18), whereas $\underset{(H)}{s}$, $\underset{\sim}{\Pi}$ and π can be eliminated with the help of appropriate state equations. Note that if the assumption (7.1) is not adopted then the equations (11.1)-(11.2) are still valid but $\underset{(H)}{s}$ occuring in these equations has to be replaced by total entropy s. One can also derive the equation (11.2) from (11.1) by using the thermodynamic identities (8.7), (8.9) and elasticity relations of the type (8.14)-(8.15).

Under assumption (7.1) we can present the equations for temperature still in other alternative forms, noting that (cf. (1.27)-(1.28)$_{2,3}$, (7.10))

$$-\left[\underset{\sim}{\Pi}\cdot\dot{\underset{\sim}{\beta}} + \pi\dot{b} + \rho T\frac{\partial \underset{(H)}{s}}{\partial \underset{\sim}{\beta}}\cdot\dot{\underset{\sim}{\beta}} + \rho T\frac{\partial \underset{(H)}{s}}{\partial b}\dot{b}\right] = -\rho\left[\frac{\partial \underset{(H)}{u}}{\partial \underset{\sim}{\beta}}\cdot\dot{\underset{\sim}{\beta}} + \frac{\partial \underset{(H)}{u}}{\partial b}\dot{b}\right] \qquad (11.4)$$

Substituting (11.4) into (11.1) and (11.2) we finally obtain

$$\begin{aligned} \varrho c_{\varepsilon} \dot{T} &= \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}^{(i)} - T \dot{\underset{\sim}{\varepsilon}}^{(e)} \cdot \underset{\sim}{L} \underset{\sim}{\alpha} - \varrho \left[\frac{\partial u_{(H)}}{\partial \underset{\sim}{\beta}} \cdot \dot{\underset{\sim}{\beta}} + \frac{\partial u_{(H)}}{\partial b} \dot{b} \right] - div\, \underset{\sim}{q} \\ \varrho c_{\sigma} \dot{T} &= \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}^{(i)} - T \underset{\sim}{\alpha} \cdot \dot{\underset{\sim}{\sigma}} - \varrho \left[\frac{\partial u_{(H)}}{\partial \underset{\sim}{\beta}} \cdot \dot{\underset{\sim}{\beta}} + \frac{\partial u_{(H)}}{\partial b} \dot{b} \right] - div\, \underset{\sim}{q} \end{aligned} \qquad (11.5)$$

In the equations (11.5) the entropy function does not occur explicitly. However, the difficulties related with the undetermined entropy function still exist. They display at the moment when one applies the inequality $(6.17)_1$ to restrict the class of postulated equations for $\dot{\underset{\sim}{\varepsilon}}^{(i)}$.

IV KINEMATICS OF FINITE ELASTO-PLASTIC DEFORMATIONS

12. Introduction

As is well known, the infinitesimal strain tensor is not a suitable strain measure for finite deformations, since it does not vanish under finite rigid body rotations. In an attempt to carry over the thermodynamical concepts developed so far it is first of all necessary to redefine the basic kinematic quantities like strain and stretching as well as their decompositions into elastic and plastic parts. The elastic-plastic body is considered as a material with an "intermediate configuration". This seems to be the most fruitful concept when dealing with finite deformations of metals. What is presented here is a concise presentation and comparison of various original ideas contained in the current literature on kinematics of finite elastic-plastic deformations. We mention especially the work of Lehmann, Lee, Sidoroff and Mandel. We have worked out in detail the relations between the coordinate-free or absolute notation and a representation in convected coordinates. It is hoped that this will be a help for the student to find his way through the literature.

13. Deformation Gradients

13.1. The Intermediate Configuration, Definition of Deformation Gradients

We consider a material body which in its natural stress-free configuration at temperature θ_0 occupies the region $\mathcal{B}^\circ$ in space at time t_0 . During a motion its material particles are carried to various positions in space. Points in space can be identified by their cartesian coordinates with respect to some fixed cartesian frame of reference with orthonormal base vectors $\underset{\sim}{e}_\alpha$.

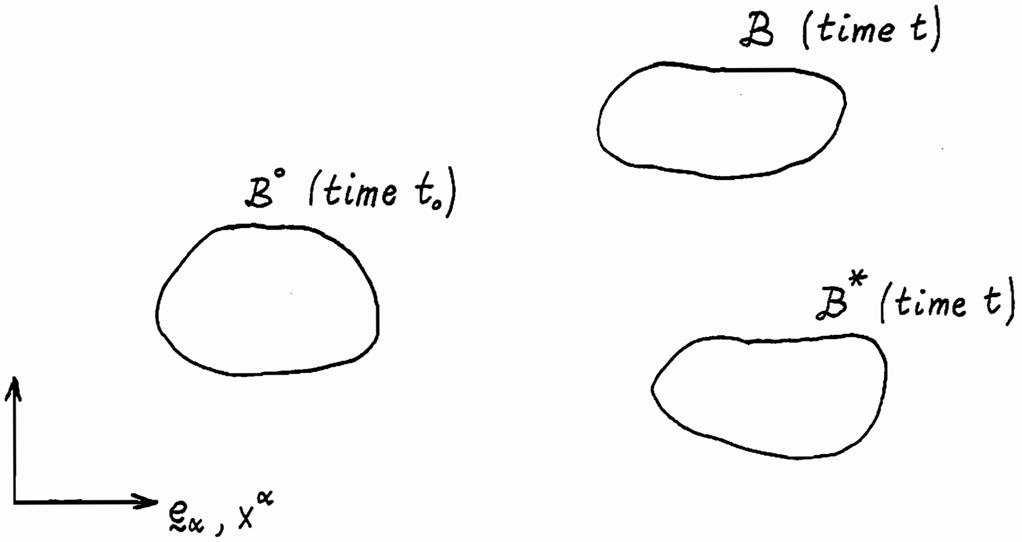

Fig. 8. Initial ($\mathcal{B}^\circ$), actual ($\mathcal{B}$) and intermediate ($\mathcal{B}^*$) configuration

To identify material points we can use their cartesian coordinates at time t_0 , denoted by $\overset{\circ}{x}{}^\alpha$. A motion is then prescribed by

$$x^\alpha = x^\alpha(\overset{\circ}{x}{}^\beta, t) \qquad (13.1)$$

where x^α are the cartesian coordinates of the material particle at time t which was initially (at time t_o) at $\overset{\circ}{x}{}^\alpha$. The set of all points $x^\alpha(\overset{\circ}{x}{}^\beta, t)$, $\overset{\circ}{x}{}^\beta \in \overset{\circ}{\mathcal{B}}$ define the actual configuration $\mathcal{B}$, a region in space, occupied by the body at time t.

In addition, we let correspond to $\mathcal{B}$ at each instant of time t a third configuration $\overset{*}{\mathcal{B}}$ which will be called intermediate or unloaded configuration

$$\overset{*}{x}{}^\alpha = \overset{*}{x}{}^\alpha(\overset{\circ}{x}{}^\beta, t) \tag{13.2}$$

This configuration would be obtained if the deformed body is instantaneously unloaded in a process of purely elastic deformations where all material elements become unstressed and are reduced to the reference temperature θ_o. A comparison between the initial configuration $\overset{\circ}{\mathcal{B}}$ and the intermediate configuration $\overset{*}{\mathcal{B}}$ then reveals the plastic part of the total deformation from $\overset{\circ}{\mathcal{B}}$ to $\mathcal{B}$. For the purpose of our study it is sufficient to consider homogeneous deformations where all material elements undergo the same deformations, stresses and temperature changes. The stress free configuration $\overset{*}{\mathcal{B}}$ is then obtained by reduction to the temperature θ_o and removal of the surface tractions in $\mathcal{B}$. In the case of inhomogeneous deformations residual stresses will remain if the body was plastically deformed and the surface tractions are removed. To remove the residual stress distribution the body has to be considered dissected into small elements which undergo different reductions in shape. As a result, the intermediate configuration will be incompatible. Also in this case the following analysis remains valid if (13.2) is interpreted as a local mapping which gives the image of an infinitesimal material element.

Let $d\overset{\circ}{\underset{\sim}{x}}$, $d\underset{\sim}{x}$, $d\overset{*}{\underset{\sim}{x}}$ be infinitesimal vectors connecting the same material particles in $\overset{\circ}{\mathcal{B}}$, $\mathcal{B}$ and $\mathcal{B}^*$. We then have

$$\begin{aligned} d\underset{\sim}{x} &= dx^{\alpha}\underset{\sim}{e}_{\alpha} = \frac{\partial x^{\alpha}}{\partial \mathring{x}^{\beta}}\, d\mathring{x}^{\beta}\underset{\sim}{e}_{\alpha} = \frac{\partial x^{\alpha}}{\partial \mathring{x}^{\beta}}\, \underset{\sim}{e}_{\alpha}\left(\underset{\sim}{e}_{\beta}\cdot d\underset{\sim}{\mathring{x}}\right) \\ &= \frac{\partial x^{\alpha}}{\partial \mathring{x}^{\beta}}\left(\underset{\sim}{e}_{\alpha}\otimes \underset{\sim}{e}_{\beta}\right) d\underset{\sim}{\mathring{x}} \qquad (13.3) \\ &= \underset{\sim}{F}\, d\underset{\sim}{\mathring{x}} \end{aligned}$$

and

$$\begin{aligned} d\underset{\sim}{\overset{*}{x}} &= d\overset{*}{x}{}^{\alpha}\underset{\sim}{e}_{\alpha} = \frac{\partial \overset{*}{x}{}^{\alpha}}{\partial \mathring{x}^{\beta}}\, d\mathring{x}^{\beta}\underset{\sim}{e}_{\alpha} = \frac{\partial \overset{*}{x}{}^{\alpha}}{\partial \mathring{x}^{\beta}}\, \underset{\sim}{e}_{\alpha}\left(\underset{\sim}{e}_{\beta}\cdot d\underset{\sim}{\mathring{x}}\right) \\ &= \frac{\partial \overset{*}{x}{}^{\alpha}}{\partial \mathring{x}^{\beta}}\left(\underset{\sim}{e}_{\alpha}\otimes \underset{\sim}{e}_{\beta}\right) d\underset{\sim}{\mathring{x}} \qquad (13.4) \\ &= \underset{\sim}{F}_{P}\, d\underset{\sim}{\mathring{x}} \end{aligned}$$

The linear transformations or second order tensors

$$\underset{\sim}{F} = \frac{\partial x^{\alpha}}{\partial \mathring{x}^{\beta}}\, \underset{\sim}{e}_{\alpha}\otimes \underset{\sim}{e}_{\beta} \qquad (13.5)$$

and

$$\underset{\sim}{F}_{P} = \frac{\partial \overset{*}{x}{}^{\alpha}}{\partial \mathring{x}^{\beta}}\, \underset{\sim}{e}_{\alpha}\otimes \underset{\sim}{e}_{\beta} \qquad (13.6)$$

relating the infinitesimal neighbourhood of a material particle in the three configurations are the (total) deformation gradient and the plastic deformation gradient*). Since the correspondence between material particles in $\mathcal{B}$, $\mathcal{B}^{\circ}$ and $\mathcal{B}^{*}$ is one to one the linear relations (13.3) and (13.4) are invertible. Hence we have

---

*) Since we consider homogeneous deformations the gradients $\underset{\sim}{F}$, $\underset{\sim}{F}_{P}$ are the same for all material points, i. e. $\frac{\partial x^{\alpha}}{\partial \mathring{x}^{\beta}}$ and $\frac{\partial \overset{*}{x}{}^{\alpha}}{\partial \mathring{x}^{\beta}}$ independent of $\mathring{x}^{\beta}$. In this case the linear relations (13.3) and (13.4) hold for vectors between any two material particles, not necessarily infinitesimal vectors.

$$d\underset{\sim}{x} = \underset{\sim}{F}\, d\underset{\sim}{\overset{\circ}{x}} = \underset{\sim}{F}\,\underset{\sim}{F}_p^{-1}\, d\underset{\sim}{\overset{*}{x}} = \underset{\sim}{F}_e\, d\underset{\sim}{\overset{*}{x}} \tag{13.7}$$

where

$$\underset{\sim}{F}_e = \underset{\sim}{F}\,\underset{\sim}{F}_p^{-1} = \frac{\partial x^{\alpha}}{\partial \overset{\circ}{x}{}^{\beta}} (\underset{\sim}{e}_{\alpha} \otimes \underset{\sim}{e}_{\beta}) \frac{\partial \overset{\circ}{x}{}^{\gamma}}{\partial \overset{*}{x}{}^{\delta}} (\underset{\sim}{e}_{\gamma} \otimes \underset{\sim}{e}_{\delta}) = \frac{\partial x^{\alpha}}{\partial \overset{\circ}{x}{}^{\gamma}} \frac{\partial \overset{\circ}{x}{}^{\gamma}}{\partial \overset{*}{x}{}^{\beta}}\, \underset{\sim}{e}_{\alpha} \otimes \underset{\sim}{e}_{\beta} \tag{13.8}$$

describes the elastic transition from $\mathcal{B}^*$ to $\mathcal{B}$.

It follows from (13.8) that the total transformation $\underset{\sim}{F}$ is multiplicatively decomposed into a plastic and elastic transformation

$$\underset{\sim}{F} = \underset{\sim}{F}_e\, \underset{\sim}{F}_p \tag{13.9}$$

Thus, $\underset{\sim}{F}$ can be interpreted as the result of a plastic transformation from the initial configuration to the intermediate configuration followed by a purely elastic transformation from the intermediate to the actual configuration. This concept is now used by many authors, e. g. Lee [31], Lehmann [32], Mandel [13], Sidoroff [33]. Following Halphen [15] it dates back to an idea of Eckart [30] as early as 1948.

Note, that the intermediate configuration is defined only up to an arbitrary rotation and therefore the decomposition (13.9) is not unique. Indeed, if the intermediate configuration is rotated by $\underset{\sim}{Q}^*$, $\underset{\sim}{Q}^*$ being orthogonal $(\underset{\sim}{Q}^{*T} = \underset{\sim}{Q}^{*-1})$ it follows

$$\underset{\sim}{F} = \underset{\sim}{F}_e\, \underset{\sim}{Q}^{*T} \underset{\sim}{Q}^{*} \underset{\sim}{F}_p = \underset{\sim}{\bar{F}}_e\, \underset{\sim}{\bar{F}}_p$$

where

$$\underset{\sim}{\bar{F}}_e = \underset{\sim}{F}_e\, \underset{\sim}{Q}^{*T} ; \quad \underset{\sim}{F}_p = \underset{\sim}{Q}^{*} \underset{\sim}{F}_p$$

yield another decomposition. The behaviour of kinematic quantities under rotations of the intermediate (and actual) configurations is needed for the discussion of invariance requirements of constitutive equations. These invariance requirements are automatically satisfied if all tensors are related to base vectors of a convected coordinate system. Before we introduce the convected coordinate system we discuss shortly curvilinear spatial coordinate systems in order to make clear the difference.

13.2. Curvilinear Spatial Coordinate Systems

If invertible functions

$$x^{\alpha} = f^{\alpha}(y^1, y^2, y^3) \tag{13.10}$$

are given, we can use both y^{α} and x^{α} for the identification of points in space. Keeping two of the three numbers y^1, y^2, y^3 fixed and changing the remaining one, say y^{β}, equation (13.10) defines coordinate lines y^{β} of a new, generally curvilinear, coordinate system. The base vectors to these coordinate lines y^{β} are

$$\underset{\sim}{G}_{\beta} = \frac{\partial f^{\alpha}}{\partial y^{\beta}} \underset{\sim}{e}_{\alpha} \tag{13.11}$$

As before, the set of points $\{\overset{\circ}{y}{}^{\alpha}\}$, $\{y^{\alpha}\}$ and $\{\overset{*}{y}{}^{\alpha}\}$ define the spatial regions occupied by the body in the initial, actual and intermediate configuration, i. e.

$$\overset{\circ}{x}{}^{\alpha} = f^{\alpha}(\overset{\circ}{y}{}^{\beta}) \; ; \qquad x^{\alpha} = f^{\alpha}(y^{\beta}) \; ; \qquad \overset{*}{x}{}^{\alpha} = f^{\alpha}(\overset{*}{y}{}^{\beta}) \quad {}^{*)} \tag{13.12}$$

---

*) Note that we have used x^{α} and y^{α} in two meanings. In (13.10) they denote an arbitrary space point, in (13.12) only those points, which lie in the region $\mathcal{B}$.

Similary $\mathring{\underset{\sim}{G}}_\alpha$, $\underset{\sim}{G}_\alpha$, $\overset{*}{\underset{\sim}{G}}_\alpha$ are base vectors of the same coordinate systems in regions $\mathring{B}$, B and $\overset{*}{B}$, respectively (see fig. 9).

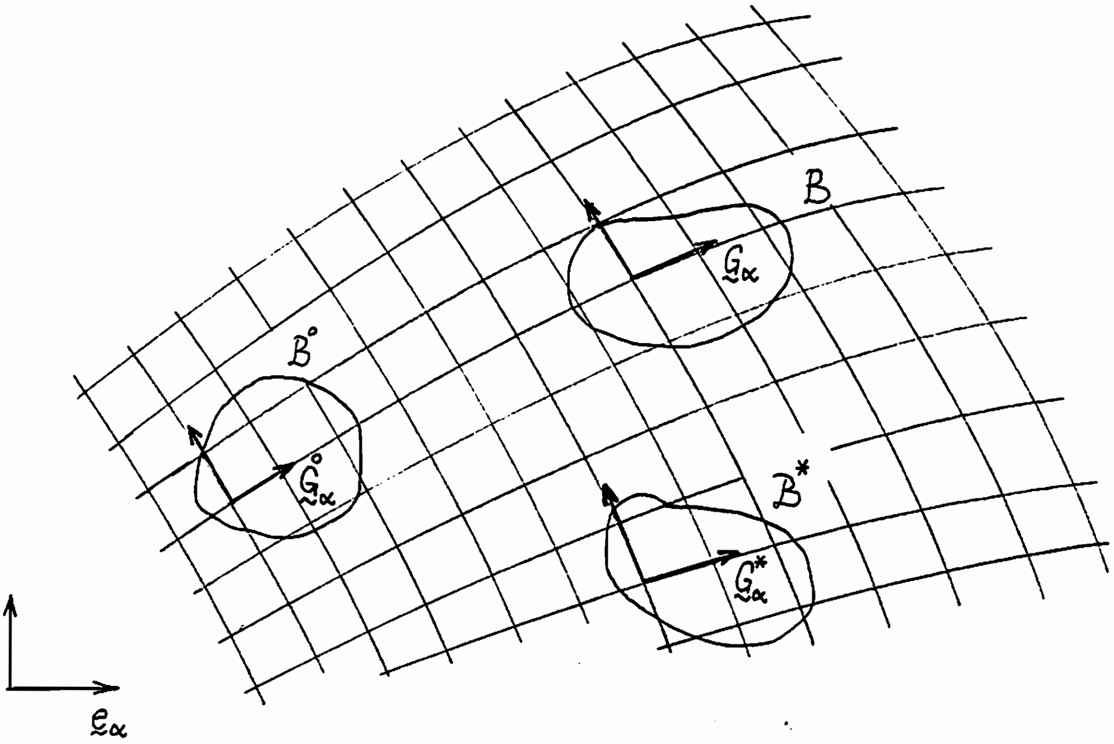

Fig. 9. Curvilinear coordinate system in space

By use of (13.12) the cartesian coordinates can be expressed by the curvilinear coordinates and (13.1), (13.2) give

$$y^\alpha = y^\alpha(\mathring{y}^\beta, t) \tag{13.13}$$

$$\overset{*}{y}^\alpha = \overset{*}{y}^\alpha(\mathring{y}^\beta, t) \tag{13.14}$$

From

$$d\underset{\sim}{x} = dx^\alpha \underset{\sim}{e}_\alpha = \frac{\partial f^\alpha}{\partial y^\beta} dy^\beta \underset{\sim}{e}_\alpha = dy^\beta \underset{\sim}{G}_\beta = \frac{\partial y^\beta}{\partial \mathring{y}^\gamma} d\mathring{y}^\gamma \underset{\sim}{G}_\beta \tag{13.15}$$

$$d\mathring{\underset{\sim}{x}} = d\mathring{x}^\alpha \underset{\sim}{e}_\alpha = \frac{\partial f^\alpha}{\partial \mathring{y}^\gamma} d\mathring{y}^\gamma \underset{\sim}{e}_\alpha = d\mathring{y}^\gamma \mathring{\underset{\sim}{G}}_\gamma \tag{13.16}$$

$$d\overset{*}{\underset{\sim}{x}} = d\overset{*}{x}^\alpha \underset{\sim}{e}_\alpha = \frac{\partial f^\alpha}{\partial \overset{*}{y}^\beta} d\overset{*}{y}^\beta \underset{\sim}{e}_\alpha = d\overset{*}{y}^\beta \overset{*}{\underset{\sim}{G}}_\beta = \frac{\partial \overset{*}{y}^\beta}{\partial \mathring{y}^\gamma} d\mathring{y}^\gamma \overset{*}{\underset{\sim}{G}}_\beta \tag{13.17}$$

and

$$d\underset{\sim}{x} = \underset{\sim}{F}\, d\underset{\sim}{\mathring{x}}\ ; \quad d\underset{\sim}{\overset{*}{x}} = \underset{\sim}{F}_p\, d\underset{\sim}{\mathring{x}}\ ; \quad d\underset{\sim}{x} = \underset{\sim}{F}_e\, d\underset{\sim}{\overset{*}{x}} \tag{13.18}$$

we get as representations for the linear transformations

$$\underset{\sim}{F} = \frac{\partial y^{\alpha}}{\partial \mathring{y}^{\beta}}\, \underset{\sim}{G}_{\alpha} \otimes \underset{\sim}{\mathring{G}}{}^{\beta} \tag{13.19}$$

$$\underset{\sim}{F}_p = \frac{\partial \overset{*}{y}{}^{\alpha}}{\partial \mathring{y}^{\beta}}\, \underset{\sim}{\overset{*}{G}}_{\alpha} \otimes \underset{\sim}{\mathring{G}}{}^{\beta} \tag{13.20}$$

$$\underset{\sim}{F}_e = \frac{\partial y^{\alpha}}{\partial \mathring{y}^{\gamma}}\, \frac{\partial \mathring{y}^{\gamma}}{\partial \overset{*}{y}{}^{\beta}}\, \underset{\sim}{G}_{\alpha} \otimes \underset{\sim}{\overset{*}{G}}{}^{\beta} \tag{13.21}$$

As example, we verify (13.19):

$$\underset{\sim}{F}\, d\underset{\sim}{\mathring{x}} = \frac{\partial y^{\alpha}}{\partial \mathring{y}^{\beta}} \left(\underset{\sim}{G}_{\alpha} \otimes \underset{\sim}{\mathring{G}}{}^{\beta}\right) d\mathring{y}^{\gamma}\, \underset{\sim}{\mathring{G}}_{\gamma} = \frac{\partial y^{\alpha}}{\partial \mathring{y}^{\beta}}\, \underset{\sim}{G}_{\alpha}\, d\mathring{y}^{\gamma} \left(\underset{\sim}{\mathring{G}}{}^{\beta} \cdot \underset{\sim}{\mathring{G}}_{\gamma}\right)$$

$$= \frac{\partial y^{\alpha}}{\partial \mathring{y}^{\beta}}\, \underset{\sim}{G}_{\alpha}\, d\mathring{y}^{\gamma}\, \delta^{\beta}_{\gamma} = \frac{\partial y^{\alpha}}{\partial \mathring{y}^{\gamma}}\, d\mathring{y}^{\gamma}\, \underset{\sim}{G}_{\alpha} = d\underset{\sim}{x}$$

In (13.21) $\dfrac{\partial \mathring{y}^{\gamma}}{\partial \overset{*}{y}{}^{\beta}}$ is the inverse of $\dfrac{\partial \overset{*}{y}{}^{\gamma}}{\partial \mathring{y}^{\beta}}$.

The matrices of the deformation gradients are now related to base vectors of different points in space and therefor transform like two-point tensor fields.

13.3. Convected Coordinates

For the identification of material particles we choose three numbers ξ^i in one-to-one correspondence with the cartesian coordinates of material points in $\mathring{\mathcal{B}}$, i. e.

$$\mathring{x}^{\alpha} = \mathring{x}^{\alpha}(\xi^i) \tag{13.22}$$

Inserting this in (13.1) and (13.2) the motion is given by

$$x^{\alpha} = x^{\alpha}(\xi^{i}, t) \tag{13.23}$$

$$\overset{*}{x}^{\alpha} = \overset{*}{x}^{\alpha}(\xi^{i}, t) \tag{13.24}$$

(13.22) defines a net of coordinate lines ξ^{i} in the initial configuration with base vectors

$$\overset{\circ}{\underset{\sim}{g}}_{i} = \frac{\partial \overset{\circ}{x}^{\alpha}}{\partial \xi^{i}} \underset{\sim}{e}_{\alpha} \tag{13.25}$$

which is deformed together with the body according to (13.23) and (13.24), respectively. The ξ^{i} are therefore called convected coordinates. Note, that the curvilinear coordinate net ξ^{i} is defined only in those regions of space that are occupied by body.

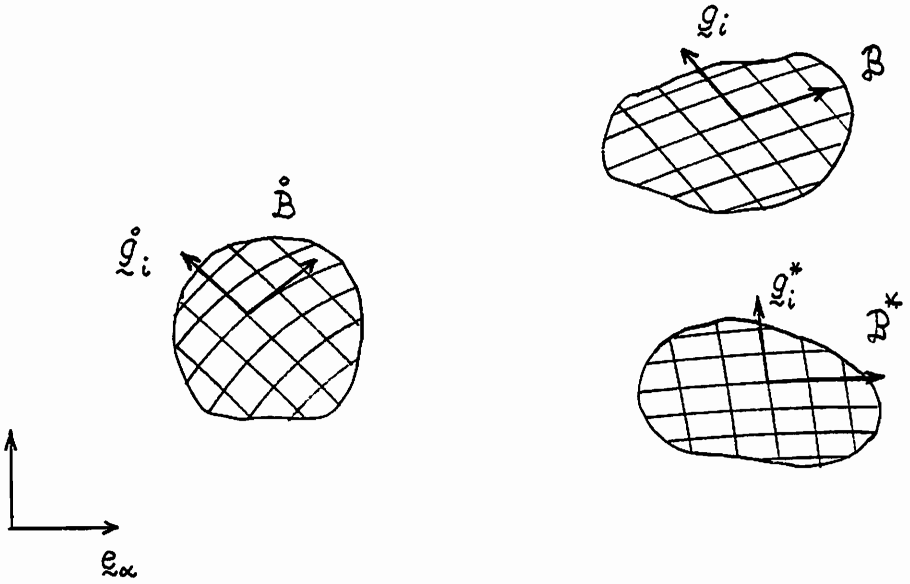

Fig. 10. Convected coordinate system

The base vectors to a coordinate line ξ^{i} , where $\xi^{k} (k \neq i)$ are constant, are time dependent and given by

$$\underset{\sim}{g}_{i} = \frac{\partial x^{\alpha}}{\partial \xi^{i}} \underset{\sim}{e}_{\alpha} \quad \text{in } \mathcal{B} \tag{13.26}$$

$$\overset{*}{\underset{\sim}{g}}_{i} = \frac{\partial \overset{*}{x}^{\alpha}}{\partial \xi^{i}} \underset{\sim}{e}_{\alpha} \quad \text{in } \overset{*}{\mathcal{B}} \tag{13.27}$$

Again, from the coordinate free relations for infinitesimal vectors between material points in $\overset{\circ}{\mathcal{B}}$, $\mathcal{B}$ and $\overset{*}{\mathcal{B}}$

$$d\underset{\sim}{x} = \underset{\sim}{F}\, d\overset{\circ}{\underset{\sim}{x}}\,; \quad d\overset{*}{\underset{\sim}{x}} = \underset{\sim}{F}_p\, d\overset{\circ}{\underset{\sim}{x}}\,; \quad d\underset{\sim}{x} = \underset{\sim}{F}_e\, d\overset{*}{\underset{\sim}{x}} \tag{13.28}$$

we obtain easily representations for the deformation gradients. E. g.

$$d\underset{\sim}{x} = dx^{\alpha}\underset{\sim}{e}_{\alpha} = \frac{\partial x^{\alpha}}{\partial \xi^{i}}\, d\xi^{i}\underset{\sim}{e}_{\alpha} = d\xi^{i}\underset{\sim}{g}_{i}$$

$$= \underset{\sim}{F}\, d\overset{\circ}{\underset{\sim}{x}} = \underset{\sim}{F}\, d\overset{\circ}{x}{}^{\alpha}\underset{\sim}{e}_{\alpha} = \underset{\sim}{F}\,\frac{\partial \overset{\circ}{x}{}^{\alpha}}{\partial \xi^{i}}\, d\xi^{i}\underset{\sim}{e}_{\alpha} = d\xi^{i}\underset{\sim}{F}\overset{\circ}{\underset{\sim}{g}}_{i}$$

hence, the base vectors at the same material particle in $\overset{\circ}{\mathcal{B}}$ and $\mathcal{B}$ obey

$$\underset{\sim}{g}_{i} = \underset{\sim}{F}\overset{\circ}{\underset{\sim}{g}}_{i} \tag{13.29}$$

This gives the representation of $\underset{\sim}{F}$ as

$$\underset{\sim}{F} = \delta^{i}_{k}\, \underset{\sim}{g}_{i} \otimes \overset{\circ}{\underset{\sim}{g}}{}^{k} = \underset{\sim}{g}_{i} \otimes \overset{\circ}{\underset{\sim}{g}}{}^{i}. \tag{13.30}$$

Indeed, $\underset{\sim}{F}\overset{\circ}{\underset{\sim}{g}}_{i} = (\underset{\sim}{g}_{k} \otimes \overset{\circ}{\underset{\sim}{g}}{}^{k})\overset{\circ}{\underset{\sim}{g}}_{i} = \underset{\sim}{g}_{k}\,(\overset{\circ}{\underset{\sim}{g}}{}^{k} \cdot \overset{\circ}{\underset{\sim}{g}}_{i}) = \underset{\sim}{g}_{i}$.

From $\overset{\circ}{\underset{\sim}{g}}_{i} = \underset{\sim}{F}^{-1}\underset{\sim}{g}_{i}$ the inverse transformation follows as

$$\underset{\sim}{F}^{-1} = \overset{\circ}{\underset{\sim}{g}}_{i} \otimes \underset{\sim}{g}^{i}. \tag{13.31}$$

The transpose of $\underset{\sim}{F}$ and $\underset{\sim}{F}^{-1}$ are

$$\underset{\sim}{F}^{T} = \overset{\circ}{\underset{\sim}{g}}{}^{i} \otimes \underset{\sim}{g}_{i} \tag{13.32}$$

$$(\underset{\sim}{F}^{-1})^{T} = \underset{\sim}{F}^{-T} = \underset{\sim}{g}^{i} \otimes \overset{\circ}{\underset{\sim}{g}}_{i} \quad {}^{*)} \tag{13.33}$$

---

*) Since $(\underset{\sim}{F}^{-1})^{T} = (\underset{\sim}{F}^{T})^{-1}$ we write for simplicity $\underset{\sim}{F}^{-T}$.

$\underset{\sim}{F}^T$ and $\underset{\sim}{F}^{-T}$ yield the contravariant base vectors $\overset{\circ}{\underset{\sim}{g}}{}^i$ and $\underset{\sim}{g}^i$ when operating on $\underset{\sim}{g}^i$ and $\overset{\circ}{\underset{\sim}{g}}{}^i$, respectively

$$\overset{\circ}{\underset{\sim}{g}}{}^i = \underset{\sim}{F}^T \underset{\sim}{g}^i \tag{13.34}$$

$$\underset{\sim}{g}^i = \underset{\sim}{F}^{-T} \overset{\circ}{\underset{\sim}{g}}{}^i \tag{13.35}$$

In the same manner representations for $\underset{\sim}{F}_e$ and $\underset{\sim}{F}_p$ are obtained. For future reference the results are listed below.

	$\underset{\sim}{F}$	$\underset{\sim}{F}_p$	$\underset{\sim}{F}_e$
relations between base vectors	$\underset{\sim}{g}_i = \underset{\sim}{F}\, \overset{\circ}{\underset{\sim}{g}}_i$ $\overset{\circ}{\underset{\sim}{g}}{}^i = \underset{\sim}{F}^T \underset{\sim}{g}^i$ $\overset{\circ}{\underset{\sim}{g}}_i = \underset{\sim}{F}^{-1} \underset{\sim}{g}_i$ $\underset{\sim}{g}^i = \underset{\sim}{F}^{-T} \overset{\circ}{\underset{\sim}{g}}{}^i$	$\overset{*}{\underset{\sim}{g}}_i = \underset{\sim}{F}_p \overset{\circ}{\underset{\sim}{g}}_i$ $\overset{\circ}{\underset{\sim}{g}}{}^i = \underset{\sim}{F}_p^T \overset{*}{\underset{\sim}{g}}{}^i$ $\overset{\circ}{\underset{\sim}{g}}_i = \underset{\sim}{F}_p^{-1} \overset{*}{\underset{\sim}{g}}_i$ $\overset{*}{\underset{\sim}{g}}{}^i = \underset{\sim}{F}_p^{-T} \overset{\circ}{\underset{\sim}{g}}{}^i$	$\underset{\sim}{g}_i = \underset{\sim}{F}_e \overset{*}{\underset{\sim}{g}}_i$ $\overset{*}{\underset{\sim}{g}}{}^i = \underset{\sim}{F}_e^T \underset{\sim}{g}^i$ $\overset{*}{\underset{\sim}{g}}_i = \underset{\sim}{F}_e^{-1} \underset{\sim}{g}_i$ $\underset{\sim}{g}^i = \underset{\sim}{F}_e^{-T} \overset{*}{\underset{\sim}{g}}{}^i .$
representations	$\underset{\sim}{F} = \underset{\sim}{g}_i \otimes \overset{\circ}{\underset{\sim}{g}}{}^i$ $\underset{\sim}{F}^T = \overset{\circ}{\underset{\sim}{g}}{}^i \otimes \underset{\sim}{g}_i$ $\underset{\sim}{F}^{-1} = \overset{\circ}{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}^i$ $\underset{\sim}{F}^{-T} = \underset{\sim}{g}^i \otimes \overset{\circ}{\underset{\sim}{g}}_i$	$\underset{\sim}{F}_p = \overset{*}{\underset{\sim}{g}}_i \otimes \overset{\circ}{\underset{\sim}{g}}{}^i$ $\underset{\sim}{F}_p^T = \overset{\circ}{\underset{\sim}{g}}{}^i \otimes \overset{*}{\underset{\sim}{g}}_i$ $\underset{\sim}{F}_p^{-1} = \overset{\circ}{\underset{\sim}{g}}_i \otimes \overset{*}{\underset{\sim}{g}}{}^i$ $\underset{\sim}{F}_p^{-T} = \overset{*}{\underset{\sim}{g}}{}^i \otimes \overset{\circ}{\underset{\sim}{g}}_i$	$\underset{\sim}{F}_e = \underset{\sim}{g}_i \otimes \overset{*}{\underset{\sim}{g}}{}^i$ $\underset{\sim}{F}_e^T = \overset{*}{\underset{\sim}{g}}{}^i \otimes \underset{\sim}{g}_i$ $\underset{\sim}{F}_e^{-1} = \overset{*}{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}^i$ $\underset{\sim}{F}_e^{-T} = \underset{\sim}{g}^i \otimes \overset{*}{\underset{\sim}{g}}_i$

Table (13.36)

The use of base vectors of a convected coordinate system leads to particularly simple forms for the matrices of deformation gradients, which become unity. The dependence of $\underset{\sim}{F}$, $\underset{\sim}{F}_e$, $\underset{\sim}{F}_p$ on the material coordinates (in the case of inhomogeneous deformations) and on time is now incorporated in the base vectors.

13.4. Behaviour under Rigid Body Rotations

If the (homogeneously) deformed body is rotated in the actual configuration $d\underset{\sim}{x}$ goes into $\overline{d\underset{\sim}{x}}$

$$\overline{d\underset{\sim}{x}} = \underset{\sim}{Q}\, d\underset{\sim}{x} \tag{13.37}$$

where $\underset{\sim}{Q}$ is proper orthogonal $(\underset{\sim}{Q}^T = \underset{\sim}{Q}^{-1}, \det \underset{\sim}{Q} = 1)$. Rotation of the intermediate configuration transforms $d\overset{*}{\underset{\sim}{x}}$ into $\overline{d\overset{*}{\underset{\sim}{x}}}$

$$\overline{d\overset{*}{\underset{\sim}{x}}} = \underset{\sim}{Q}^{*}\, d\overset{*}{\underset{\sim}{x}} \qquad (\underset{\sim}{Q}^{*T} = \underset{\sim}{Q}^{*-1},\ \det Q^{*} = 1) \tag{13.38}$$

Since $d\overset{\circ}{\underset{\sim}{x}}$ remains unchanged,

$$\overline{d\overset{\circ}{\underset{\sim}{x}}} = d\overset{\circ}{\underset{\sim}{x}} \tag{13.39}$$

the transformation behaviour of the deformation gradients is obtained from

$$\overline{d\underset{\sim}{x}} = \overline{\underset{\sim}{F}}\, \overline{d\overset{\circ}{\underset{\sim}{x}}}\ , \quad \overline{d\underset{\sim}{x}} = \overline{\underset{\sim}{F}_e}\, \overline{d\overset{*}{\underset{\sim}{x}}}\ , \quad \overline{d\overset{*}{\underset{\sim}{x}}} = \overline{\underset{\sim}{F}_p}\, \overline{d\overset{\circ}{\underset{\sim}{x}}}$$

as

$$\overline{\underset{\sim}{F}} = \underset{\sim}{Q}\,\underset{\sim}{F} \tag{13.40}$$

$$\overline{\underset{\sim}{F}_p} = \underset{\sim}{Q}^{*}\,\underset{\sim}{F}_p \tag{13.41}$$

$$\overline{\underset{\sim}{F}_e} = \underset{\sim}{Q}\,\underset{\sim}{F}_e\,\underset{\sim}{Q}^{*T} \tag{13.42}$$

14. Strain

14.1. Definition of Strain Measures and their Decomposition

With the help of the linear transformations $\underset{\sim}{F}$, $\underset{\sim}{F}_e$ and $\underset{\sim}{F}_p$ which relate vectors between two material points in the initial, actual and intermediate configuration

$$
\begin{aligned}
d\underset{\sim}{x} &= \underset{\sim}{F}\, d\overset{\circ}{\underset{\sim}{x}} \\
d\overset{*}{\underset{\sim}{x}} &= \underset{\sim}{F}_p\, d\overset{\circ}{\underset{\sim}{x}} \qquad\qquad (14.1) \\
d\underset{\sim}{x} &= \underset{\sim}{F}_e\, d\overset{*}{\underset{\sim}{x}}
\end{aligned}
$$

we can easily calculate the square of the arc lengths in $\overset{\circ}{\mathcal{B}}$, $\mathcal{B}$ and $\mathcal{B}^*$.

For example,

$$
\begin{aligned}
d\overset{*}{s}{}^2 = (d\overset{*}{s})^2 &= d\overset{*}{\underset{\sim}{x}} \cdot d\overset{*}{\underset{\sim}{x}} \\
&= \underset{\sim}{F}_p d\overset{\circ}{\underset{\sim}{x}} \cdot \underset{\sim}{F}_p d\overset{\circ}{\underset{\sim}{x}} = d\overset{\circ}{\underset{\sim}{x}} \cdot \underset{\sim}{F}_p^T \underset{\sim}{F}_p\, d\overset{\circ}{\underset{\sim}{x}} \\
&= \underset{\sim}{F}_e^{-1} d\underset{\sim}{x} \cdot \underset{\sim}{F}_e^{-1} d\underset{\sim}{x} = d\underset{\sim}{x} \cdot \underset{\sim}{F}_e^{-T} \underset{\sim}{F}_e^{-1} d\underset{\sim}{x}
\end{aligned}
$$

The result of similar calculations for ds^2 and $d\overset{\circ}{s}{}^2$ is summarized in the following table:

$ds^2 =$	$d\underset{\sim}{x} \cdot d\underset{\sim}{x}$	$d\overset{\circ}{\underset{\sim}{x}} \cdot \underset{\sim}{F}^T \underset{\sim}{F}\, d\overset{\circ}{\underset{\sim}{x}}$	$d\overset{*}{\underset{\sim}{x}} \cdot \underset{\sim}{F}_e^T \underset{\sim}{F}_e\, d\overset{*}{\underset{\sim}{x}}$
$d\overset{\circ}{s}{}^2 =$	$d\underset{\sim}{x} \cdot (\underset{\sim}{F}\underset{\sim}{F}^T)^{-1} d\underset{\sim}{x}$	$d\overset{\circ}{\underset{\sim}{x}} \cdot d\overset{\circ}{\underset{\sim}{x}}$	$d\overset{*}{\underset{\sim}{x}} \cdot (\underset{\sim}{F}_p \underset{\sim}{F}_p^T)^{-1} d\overset{*}{\underset{\sim}{x}}$
$d\overset{*}{s}{}^2 =$	$d\underset{\sim}{x} \cdot (\underset{\sim}{F}_e \underset{\sim}{F}_e^T)^{-1} d\underset{\sim}{x}$	$d\overset{\circ}{\underset{\sim}{x}} \cdot \underset{\sim}{F}_p^T \underset{\sim}{F}_p\, d\overset{\circ}{\underset{\sim}{x}}$	$d\overset{*}{\underset{\sim}{x}} \cdot d\overset{*}{\underset{\sim}{x}}$

(14.2)

↓ *strain tensors in $\mathcal{B}$ (Eulerian description)*

↓ *strain tensors in $\mathcal{B}^\circ$ (Lagrangian description)*

↓ *strain tensors in $\mathcal{B}^*$*

If we know the position of material particles in one configuration and two symmetric tensors we are able to calculate the length of line elements between material particles in the two other configurations. The symmetric tensors occuring in (14.2) are therefore basic for the definition of strain. Furthermore, we can distinguish between strain measures related to the actual, initial and intermediate configuration. They appear in the first, second and third column of the table (14.2), respectively. With the notation

$$\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F} \; ; \qquad \underset{\sim}{C}_e = \underset{\sim}{F}_e^T\underset{\sim}{F}_e \; ; \qquad \underset{\sim}{C}_p = \underset{\sim}{F}_p^T\underset{\sim}{F}_p \tag{14.3}$$

$$\underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T ; \qquad \underset{\sim}{B}_e = \underset{\sim}{F}_e\underset{\sim}{F}_e^T ; \qquad \underset{\sim}{B}_p = \underset{\sim}{F}_p\underset{\sim}{F}_p^T \tag{14.4}$$

and (14.2) we obtain for the difference of the square of the line elements

$ds^2 - d\overset{\circ}{s}^2 =$	$d\underset{\sim}{x}\cdot(\underset{\sim}{1}-\underset{\sim}{B}^{-1})d\underset{\sim}{x}$	$d\overset{\circ}{\underset{\sim}{x}}\cdot(\underset{\sim}{C}-\underset{\sim}{1})d\overset{\circ}{\underset{\sim}{x}}$	$d\overset{*}{\underset{\sim}{x}}\cdot(\underset{\sim}{C}_e-\underset{\sim}{B}_p^{-1})d\overset{*}{\underset{\sim}{x}}$
$ds^2 - d\overset{*}{s}^2 =$	$d\underset{\sim}{x}\cdot(\underset{\sim}{1}-\underset{\sim}{B}_e^{-1})d\underset{\sim}{x}$	$d\overset{\circ}{\underset{\sim}{x}}\cdot(\underset{\sim}{C}-\underset{\sim}{C}_p)d\overset{\circ}{\underset{\sim}{x}}$	$d\overset{*}{\underset{\sim}{x}}\cdot(\underset{\sim}{C}_e-\underset{\sim}{1})d\overset{*}{\underset{\sim}{x}}$
$d\overset{*}{s}^2 - d\overset{\circ}{s}^2 =$	$d\underset{\sim}{x}\cdot(\underset{\sim}{B}_e^{-1}-\underset{\sim}{B}^{-1})d\underset{\sim}{x}$	$d\overset{\circ}{\underset{\sim}{x}}\cdot(\underset{\sim}{C}_p-\underset{\sim}{1})d\overset{\circ}{\underset{\sim}{x}}$	$d\overset{*}{\underset{\sim}{x}}\cdot(\underset{\sim}{1}-\underset{\sim}{B}_p^{-1})d\overset{*}{\underset{\sim}{x}}$

(14.5)

(14.5) motivates the definition of the total, elastic and plastic strain measures according to

$2\underset{\sim}{\varepsilon} = \underset{\sim}{1}-\underset{\sim}{B}^{-1}$	$2\overset{\circ}{\underset{\sim}{\varepsilon}} = \underset{\sim}{C}-\underset{\sim}{1}$	$2\overset{*}{\underset{\sim}{\varepsilon}} = \underset{\sim}{C}_e-\underset{\sim}{B}_p^{-1}$
$2\underset{\sim}{\varepsilon}_e = \underset{\sim}{1}-\underset{\sim}{B}_e^{-1}$	$2\overset{\circ}{\underset{\sim}{\varepsilon}}_e = \underset{\sim}{C}-\underset{\sim}{C}_p$	$2\overset{*}{\underset{\sim}{\varepsilon}}_e = \underset{\sim}{C}_e-\underset{\sim}{1}$
$2\underset{\sim}{\varepsilon}_p = \underset{\sim}{B}_e^{-1}-\underset{\sim}{B}^{-1}$	$2\overset{\circ}{\underset{\sim}{\varepsilon}}_p = \underset{\sim}{C}_p-\underset{\sim}{1}$	$2\overset{*}{\underset{\sim}{\varepsilon}}_p = \underset{\sim}{1}-\underset{\sim}{B}_p^{-1}$

(14.6)

This definition of elastic and plastic strain tensors leads in all three cases to an additive decomposition of the total strain

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}_e+\underset{\sim}{\varepsilon}_p \; ; \qquad \overset{\circ}{\underset{\sim}{\varepsilon}} = \overset{\circ}{\underset{\sim}{\varepsilon}}_e+\overset{\circ}{\underset{\sim}{\varepsilon}}_p \; ; \qquad \overset{*}{\underset{\sim}{\varepsilon}} = \overset{*}{\underset{\sim}{\varepsilon}}_e+\overset{*}{\underset{\sim}{\varepsilon}}_p . \tag{14.7}$$

As can be seen from (14.6) the total, elastic and plastic strain tensors are defined in terms of

$\underset{\sim}{B}$ and $\underset{\sim}{B}_e$ in the Eulerian description,

and in terms of

$\underset{\sim}{C}$ and $\underset{\sim}{C}_p$ in the Lagrangian description.

The tensors $\underset{\sim}{\varepsilon}^*$, $\underset{\sim}{\varepsilon}_e^*$, $\underset{\sim}{\varepsilon}_p^*$, related to the intermediate configuration, are defined by $\underset{\sim}{C}_e$ and $\underset{\sim}{B}_p$. Thus the total strain $\underset{\sim}{\varepsilon}^*$ can not be expressed in a simple way by the gradient of deformation. These tensors are introduced by Sidoroff [33]. $\overset{\circ}{\underset{\sim}{\varepsilon}}$ and $\overset{\circ}{\underset{\sim}{\varepsilon}}_p$ with $\overset{\circ}{\underset{\sim}{\varepsilon}}_e$ defined as $\overset{\circ}{\underset{\sim}{\varepsilon}}_e = \overset{\circ}{\underset{\sim}{\varepsilon}} - \overset{\circ}{\underset{\sim}{\varepsilon}}_p$ are used by Green and Naghdi [34, 35, 36]
$\overset{\circ}{\underset{\sim}{\varepsilon}}$, $\overset{*}{\underset{\sim}{\varepsilon}}$ and $\underset{\sim}{\varepsilon}$ are interrelated by

$$\overset{\circ}{\underset{\sim}{\varepsilon}} = \underset{\sim}{F}^T \underset{\sim}{\varepsilon} \underset{\sim}{F} ; \qquad \overset{\circ}{\underset{\sim}{\varepsilon}} = \underset{\sim}{F}_p^T \underset{\sim}{\varepsilon}^* \underset{\sim}{F}_p ; \qquad \underset{\sim}{\varepsilon}^* = \underset{\sim}{F}_e^T \underset{\sim}{\varepsilon} \underset{\sim}{F}_e \tag{14.8}$$

with analogous formulae for their elastic and plastic components. Lehmann [32, 38] has introduced a multiplikative decomposition of the left and right Cauchy-Green tensors $\underset{\sim}{B}$ and $\underset{\sim}{C}$

$$\underset{\sim}{B} = \underset{\sim}{B}_p' \underset{\sim}{B}_e = \underset{\sim}{B}_e \underset{\sim}{B}_p'^T \quad \text{with} \quad \underset{\sim}{B}_p' = \underset{\sim}{F}_e \underset{\sim}{B}_p \underset{\sim}{F}_e^{-1} \tag{14.9}$$

$$\underset{\sim}{C} = \underset{\sim}{C}_p \underset{\sim}{C}_e' = \underset{\sim}{C}_e'^T \underset{\sim}{C}_p \quad \text{with} \quad \underset{\sim}{C}_e' = \underset{\sim}{F}_p^{-1} \underset{\sim}{C}_e \underset{\sim}{F}_p \tag{14.10}$$

Clearly,

$$\underset{\sim}{B}_p' \underset{\sim}{B}_e = \underset{\sim}{F}_e \underset{\sim}{B}_p \underset{\sim}{F}_e^{-1} \underset{\sim}{B}_e = \underset{\sim}{F}_e \underset{\sim}{F}_p \underset{\sim}{F}_p^T \underset{\sim}{F}_e^{-1} \underset{\sim}{F}_e \underset{\sim}{F}_e^T = \underset{\sim}{F}_e \underset{\sim}{F}_p (\underset{\sim}{F}_e \underset{\sim}{F}_p)^T = \underset{\sim}{B}$$

$$\underset{\sim}{C}_p \underset{\sim}{C}_e' = \underset{\sim}{C}_p \underset{\sim}{F}_p^{-1} \underset{\sim}{C}_e \underset{\sim}{F}_p = \underset{\sim}{F}_p^T \underset{\sim}{F}_p \underset{\sim}{F}_p^{-1} \underset{\sim}{F}_e^T \underset{\sim}{F}_e \underset{\sim}{F}_p = (\underset{\sim}{F}_e \underset{\sim}{F}_p)^T \underset{\sim}{F}_e \underset{\sim}{F}_p = \underset{\sim}{C}$$

14.2. Representation in Convected Coordinates

It is illustrative to give representations of the tensors defined above in convected coordinates. Since the deformation gradients can be expressed as dyadic products of base vectors of the three configurations (13.36), this is a simple exercise.

For example,

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} = (\underset{\sim}{\overset{\circ}{g}}{}^i \otimes \underset{\sim}{g}_i)(\underset{\sim}{g}_k \otimes \underset{\sim}{\overset{\circ}{g}}{}^k) = (\underset{\sim}{g}_i \cdot \underset{\sim}{g}_k)\, \underset{\sim}{\overset{\circ}{g}}{}^i \otimes \underset{\sim}{\overset{\circ}{g}}{}^k = g_{ik}\, \underset{\sim}{\overset{\circ}{g}}{}^i \otimes \underset{\sim}{\overset{\circ}{g}}{}^k$$

We obtain in the same manner

$$\begin{aligned}
&\underset{\sim}{C}_e = \underset{\sim}{F}_e^T \underset{\sim}{F}_e = g_{ik}\, \underset{\sim}{\overset{*}{g}}{}^i \otimes \underset{\sim}{\overset{*}{g}}{}^k ;\\
&\underset{\sim}{C}_p = \underset{\sim}{F}_p^T \underset{\sim}{F}_p = \overset{*}{g}_{ik}\, \underset{\sim}{\overset{\circ}{g}}{}^i \otimes \underset{\sim}{\overset{\circ}{g}}{}^k ;\\
&\underset{\sim}{C}_e' = \underset{\sim}{F}_p^{-1} \underset{\sim}{C}_e \underset{\sim}{F}_p = \overset{*}{g}{}^{ir} g_{rk}\, \underset{\sim}{\overset{\circ}{g}}_i \otimes \underset{\sim}{\overset{\circ}{g}}{}^k ;\\
&\underset{\sim}{B} = \underset{\sim}{F} \underset{\sim}{F}^T = \overset{\circ}{g}{}^{ik}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}_k ; \qquad \underset{\sim}{B}^{-1} = \overset{\circ}{g}_{ik}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k ;\\
&\underset{\sim}{B}_e = \underset{\sim}{F}_e \underset{\sim}{F}_e^T = \overset{*}{g}{}^{ik}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}_k ; \qquad \underset{\sim}{B}_e^{-1} = \overset{*}{g}_{ik}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k ;\\
&\underset{\sim}{B}_p = \underset{\sim}{F}_p \underset{\sim}{F}_p^T = \overset{\circ}{g}{}^{ik}\, \underset{\sim}{\overset{*}{g}}_i \otimes \underset{\sim}{\overset{*}{g}}_k ; \qquad \underset{\sim}{B}_p^{-1} = \overset{\circ}{g}_{ik}\, \underset{\sim}{\overset{*}{g}}{}^i \otimes \underset{\sim}{\overset{*}{g}}{}^k ;\\
&\underset{\sim}{B}_p' = \underset{\sim}{F}_e \underset{\sim}{B}_p \underset{\sim}{F}_e^{-1} = \overset{\circ}{g}{}^{ir} g_{rk}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}^k .
\end{aligned} \qquad (14.11)$$

Observing, that the unit tensor can be represented as

$$\underset{\sim}{1} = g_{ik}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k = \overset{\circ}{g}_{ik}\, \underset{\sim}{\overset{\circ}{g}}{}^i \otimes \underset{\sim}{\overset{\circ}{g}}{}^k = \overset{*}{g}_{ik}\, \underset{\sim}{\overset{*}{g}}{}^i \otimes \underset{\sim}{\overset{*}{g}}{}^k \qquad (14.12)$$

the strain measures $\underset{\sim}{\overset{\circ}{\varepsilon}}$, $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\overset{*}{\varepsilon}}$ and their elastic and plastic parts (14.6) become

$$\begin{aligned}
2\underset{\sim}{\varepsilon} &= \underset{\sim}{1} - \underset{\sim}{B}^{-1} = (g_{ik} - \overset{\circ}{g}_{ik})\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k\\
2\underset{\sim}{\varepsilon}_e &= \underset{\sim}{1} - \underset{\sim}{B}_e^{-1} = (g_{ik} - \overset{*}{g}_{ik})\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k\\
2\underset{\sim}{\varepsilon}_p &= \underset{\sim}{B}_e^{-1} - \underset{\sim}{B}^{-1} = (\overset{*}{g}_{ik} - \overset{\circ}{g}_{ik})\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k
\end{aligned} \qquad (14.13)$$

$$\begin{aligned} 2\overset{\circ}{\underset{\sim}{\varepsilon}} &= \underset{\sim}{C} - \underset{\sim}{1} = (g_{ik} - \overset{\circ}{g}_{ik})\, \overset{\circ}{\underset{\sim}{g}}{}^{i} \otimes \overset{\circ}{\underset{\sim}{g}}{}^{k}; \\ 2\overset{\circ}{\underset{\sim}{\varepsilon}}{}_{e} &= \underset{\sim}{C} - \underset{\sim}{C}_{p} = (g_{ik} - \overset{*}{g}_{ik})\, \overset{\circ}{\underset{\sim}{g}}{}^{i} \otimes \overset{\circ}{\underset{\sim}{g}}{}^{k}; \\ 2\overset{\circ}{\underset{\sim}{\varepsilon}}{}_{p} &= \underset{\sim}{C}_{p} - \underset{\sim}{1} = (\overset{*}{g}_{ik} - \overset{\circ}{g}_{ik})\, \overset{\circ}{\underset{\sim}{g}}{}^{i} \otimes \overset{\circ}{\underset{\sim}{g}}{}^{k}; \end{aligned} \qquad (14.14)$$

$$\begin{aligned} 2\overset{*}{\underset{\sim}{\varepsilon}} &= \underset{\sim}{C}_{e} - \underset{\sim}{B}_{p}^{-1} = (g_{ik} - \overset{\circ}{g}_{ik})\, \overset{*}{\underset{\sim}{g}}{}^{i} \otimes \overset{*}{\underset{\sim}{g}}{}^{k}; \\ 2\overset{*}{\underset{\sim}{\varepsilon}}{}_{e} &= \underset{\sim}{C}_{e} - \underset{\sim}{1} = (g_{ik} - \overset{*}{g}_{ik})\, \overset{*}{\underset{\sim}{g}}{}^{i} \otimes \overset{*}{\underset{\sim}{g}}{}^{k}; \\ 2\overset{*}{\underset{\sim}{\varepsilon}}{}_{p} &= \underset{\sim}{1} - \underset{\sim}{B}_{p}^{-1} = (\overset{*}{g}_{ik} - \overset{\circ}{g}_{ik})\, \overset{*}{g}{}^{i} \otimes \overset{*}{g}{}^{k}. \end{aligned} \qquad (14.15)$$

Equations (14.11)-(14.15) show the importance of the metric coefficients g_{ik}, $\overset{\circ}{g}_{ik}$, $\overset{*}{g}_{ik}$ for the description of strain*). This can be expected, of course, since the metric coefficients are directly connected with the length of line elements, e. g.

$$ds^2 = d\underset{\sim}{x} \cdot d\underset{\sim}{x} = d\xi^i \underset{\sim}{g}_i \cdot d\xi^k \underset{\sim}{g}_k = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_k\, d\xi^i d\xi^k = g_{ik}\, d\xi^i d\xi^k$$

Note further, that all strain measures $\underset{\sim}{\varepsilon}$, $\overset{\circ}{\underset{\sim}{\varepsilon}}$, $\overset{*}{\underset{\sim}{\varepsilon}}$ and their elastic and plastic parts have the same covariant components, however with respect to different base vectors. They, therefore, define different tensors.

14.3. Behaviour under Superimposed Rigid Body Rotations

Under rigid body rotations of the intermediate and actual configuration, described by the orthogonal tensors $\underset{\sim}{Q}^{*}$ and $\underset{\sim}{Q}$, respectively, the deformation gradients transform

---

*) Relating these coefficients to different base vectors all previously defined kinematic quantities can be expressed.

into (13.40)-(13.42)

$$\bar{\underset{\sim}{F}} = \underset{\sim}{Q}\,\underset{\sim}{F}\,; \quad \bar{\underset{\sim}{F}}_e = \underset{\sim}{Q}\,\underset{\sim}{F}_e\,\underset{\sim}{Q}^{*T}; \quad \bar{\underset{\sim}{F}}_p = \underset{\sim}{Q}^{*}\underset{\sim}{F}_p \qquad (14.16)$$

The behaviour of the kinematic tensors under such rotations is listed below.

Behaviour under transformation	Quantities
$\bar{\underset{\sim}{X}} = \underset{\sim}{X}$ (invariance)	$\underset{\sim}{C},\ \underset{\sim}{C}_p,\ \mathring{\underset{\sim}{\varepsilon}},\ \mathring{\underset{\sim}{\varepsilon}}_e,\ \mathring{\underset{\sim}{\varepsilon}}_p,\ \underset{\sim}{C}'_e$
$\bar{\underset{\sim}{X}} = \underset{\sim}{Q}\,\underset{\sim}{X}\,\underset{\sim}{Q}^T$	$\underset{\sim}{B},\ \underset{\sim}{B}_e,\ \underset{\sim}{\varepsilon},\ \underset{\sim}{\varepsilon}_e,\ \underset{\sim}{\varepsilon}_p,\ \underset{\sim}{B}'_p$
$\bar{\underset{\sim}{X}} = \underset{\sim}{Q}^{*}\underset{\sim}{X}\,\underset{\sim}{Q}^{*T}$	$\underset{\sim}{C}_e,\ \underset{\sim}{B}_p,\ \overset{*}{\underset{\sim}{\varepsilon}},\ \overset{*}{\underset{\sim}{\varepsilon}}_e,\ \overset{*}{\underset{\sim}{\varepsilon}}_p$

(14.17)

E. g.

$$\bar{\underset{\sim}{B}}_e = \bar{\underset{\sim}{F}}_e\bar{\underset{\sim}{F}}_e^T = \underset{\sim}{Q}\,\underset{\sim}{F}_e\,\underset{\sim}{Q}^{*T}\underset{\sim}{Q}^{*}\underset{\sim}{F}_e^T\,\underset{\sim}{Q}^T = \underset{\sim}{Q}\,\underset{\sim}{F}_e\underset{\sim}{F}_e^T\,\underset{\sim}{Q}^T = \underset{\sim}{Q}\,\underset{\sim}{B}_e\,\underset{\sim}{Q}^T.$$

14.4. Discussion

Introduction of the notion of intermediate configuration enables to present rigorous geometrical interpretation of various measures of elastic and plastic deformation, and shows how the total finite strain can be decomposed into two parts. This is the first and very fundamental step to their full operational definitions. The concept of intermediate configuration seems also to be very useful when developing the thermodynamic theory of elastic-plastic rate-independent behaviour of metals since the thermoelastic characteristics of metals are measured with respect to actual unstressed configuration no matter wheather the given piece of metal was or was not previously plastically deformed. Moreover thermo-elastic properties of metals so experimentally determined are essentially not influenced by previous plastic deformations. Thus the theory that utilizes the notion of the intermediate configuration may directly refer to the results of such measurements, and to be relatively simple. Bearing in mind this pragmatic point of view it is expedient to take as a basic external parameter of thermodynamic state such measure of elastic strains defined in sec. 14 that is completely expressed in terms of $\underset{\sim}{F}_e$ (i. e., such which is independent of $\underset{\sim}{F}_p$). For example, tensor $\underset{\sim}{E}_e^{*}$ (or $\underset{\sim}{C}_e$) may be treated as proper thermodynamic state parameter (it may be used for description of elastic properties) but not tensor $\overset{\circ}{\underset{\sim}{E}}_e$. The latter depend also on $\underset{\sim}{F}_p$ what means that knowledge of the very special (initial) configuration of a plastically deformed and unstressed piece of metal is required for determination of its thermoelastic state. There is no strong physical arguments why it should be so. Hence, pair $\overset{\circ}{\underset{\sim}{E}}_e$ and temperature can not be taken alone to describe the thermo-elastic state of previously plastically deformed material.

On the other hand, use of the tensor $\underset{\sim}{E}_e^{*}$ leads to some problems in the case of nonisotropic elastic behaviour. Under rigid body rotation of intermediate configuration tensor $\underset{\sim}{E}_e^{*}$ transforms according to $\underset{\sim}{Q}^{*}\underset{\sim}{E}_e^{*}\underset{\sim}{Q}^{*T}$. If one assumes the elastic response (with respect to the unstressed intermediate configuration) to be invariant under such rotation one gets as a conclusion that elastic behaviour must be necessary isotropic. This problem can be solved by selection of well specified orientation of an intermediate configuration. The decomposition $\underset{\sim}{F} = \underset{\sim}{F}_e\underset{\sim}{F}_p$ becames then unique. Such specification of the intermediate configuration can be done in various manner. For example Mandel [13] has introduced the concept of so called "director frame". The discussion of this concept is, however, beyond the scope of this seminar-notes. Some other possibilities are mentioned in sec. 15.

15. Rate of Deformation

15.1. Definitions and Decompositions

As in the discussion of strain our starting point is again the relation between infinitesimal vectors between two material points in the various configurations

$$d\underset{\sim}{x} = \underset{\sim}{F}\, d\overset{\circ}{\underset{\sim}{x}}\ ; \qquad d\underset{\sim}{x}^* = \underset{\sim}{F}_p\, d\overset{\circ}{\underset{\sim}{x}}; \qquad d\underset{\sim}{x} = \underset{\sim}{F}_e\, d\underset{\sim}{x}^* \tag{15.1}$$

Since $d\overset{\circ}{\underset{\sim}{x}}$ is unchanged the material time derivatives give

$$(d\underset{\sim}{x})^{\cdot} = \dot{\underset{\sim}{F}}\, d\overset{\circ}{\underset{\sim}{x}} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} d\underset{\sim}{x} = \dot{\underset{\sim}{F}}\underset{\sim}{F}_p^{-1} d\underset{\sim}{x}^* \tag{15.2}$$

$$(d\underset{\sim}{x}^*)^{\cdot} = \dot{\underset{\sim}{F}}_p\, d\overset{\circ}{\underset{\sim}{x}} = \dot{\underset{\sim}{F}}_p\underset{\sim}{F}^{-1} d\underset{\sim}{x} = \dot{\underset{\sim}{F}}_p\underset{\sim}{F}_p^{-1} d\underset{\sim}{x}^*. \tag{15.3}$$

$\dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1}$ is the spatial gradient of the velocity field and will be denoted by $\underset{\sim}{L}$, the corresponding quantity in the intermediate configuration $\dot{\underset{\sim}{F}}_p\underset{\sim}{F}_p^{-1}$ will be denoted by $\underset{\sim}{L}^*$.

$$(d\underset{\sim}{x})^{\cdot} = \underset{\sim}{L}\, d\underset{\sim}{x}\ ; \qquad \underset{\sim}{L} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} \tag{15.4}$$

$$(d\underset{\sim}{x}^*)^{\cdot} = \underset{\sim}{L}^* d\underset{\sim}{x}^*\ ; \qquad \underset{\sim}{L}^* = \dot{\underset{\sim}{F}}_p\underset{\sim}{F}_p^{-1} \tag{15.5}$$

Their symmetric and skew parts describe stretching $(\underset{\sim}{D},\underset{\sim}{D}^*)$ and spin $(\underset{\sim}{W},\underset{\sim}{W}^*)$ in the actual and intermediate configuration

$$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{W}\ ; \quad \underset{\sim}{D} = \tfrac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T) = \{\underset{\sim}{L}\}^{*)}; \quad \underset{\sim}{W} = \tfrac{1}{2}(\underset{\sim}{L} - \underset{\sim}{L}^T) \tag{15.6}$$

$$\underset{\sim}{L}^* = \underset{\sim}{D}^* + \underset{\sim}{W}^*; \quad \underset{\sim}{D}^* = \tfrac{1}{2}(\underset{\sim}{L}^* + \underset{\sim}{L}^{*T}) = \{\underset{\sim}{L}^*\}; \quad \underset{\sim}{W}^* = \tfrac{1}{2}(\underset{\sim}{L}^* - \underset{\sim}{L}^{*T}). \tag{15.7}$$

---

*) $\{\ldots\}$ denotes always the symmetric part of the tensor in brackets.

The rate of change in length of material line elements can now be expressed by $\underset{\sim}{D}$ and $\underset{\sim}{D}^*$.

$$(ds^2)^\cdot = (d\underset{\sim}{x}\cdot d\underset{\sim}{x})^\cdot = 2d\underset{\sim}{x}\cdot d\dot{\underset{\sim}{x}} = 2d\underset{\sim}{x}\cdot \underset{\sim}{L}\,d\underset{\sim}{x} = d\underset{\sim}{x}\cdot 2\underset{\sim}{D}\,d\underset{\sim}{x}; \qquad (15.8)$$

$$(ds^{*2})^\cdot = (d\overset{*}{\underset{\sim}{x}}\cdot d\overset{*}{\underset{\sim}{x}})^\cdot = 2d\overset{*}{\underset{\sim}{x}}\cdot(d\overset{*}{\underset{\sim}{x}})^\cdot = 2d\overset{*}{\underset{\sim}{x}}\cdot \overset{*}{\underset{\sim}{L}}\,d\overset{*}{\underset{\sim}{x}} = d\overset{*}{\underset{\sim}{x}}\cdot 2\underset{\sim}{D}^*d\overset{*}{\underset{\sim}{x}}, \qquad (15.9)$$

taking into account that

$$d\underset{\sim}{x}\cdot \underset{\sim}{W}d\underset{\sim}{x} = \underset{\sim}{W}^T d\underset{\sim}{x}\cdot d\underset{\sim}{x} = -\underset{\sim}{W}d\underset{\sim}{x}\cdot d\underset{\sim}{x} = 0$$

for a skew tensor $\underset{\sim}{W}$. It follows that

$$\tfrac{1}{2}(ds^2 - d\overset{\circ}{s}{}^2)^\cdot = d\underset{\sim}{x}\cdot \underset{\sim}{D}\,d\underset{\sim}{x};$$

$$\tfrac{1}{2}(ds^{*2} - d\overset{\circ}{s}{}^2)^\cdot = d\overset{*}{\underset{\sim}{x}}\cdot \underset{\sim}{D}^*d\overset{*}{\underset{\sim}{x}} = \underset{\sim}{F}_e^{-1}d\underset{\sim}{x}\cdot \underset{\sim}{D}^*\underset{\sim}{F}_e^{-1}d\underset{\sim}{x} = d\underset{\sim}{x}\cdot \underset{\sim}{F}_e^{-T}\underset{\sim}{D}^*\underset{\sim}{F}_e^{-1}d\underset{\sim}{x}; \qquad (15.10)$$

$$\tfrac{1}{2}(ds^2 - ds^{*2})^\cdot = d\underset{\sim}{x}\cdot \underset{\sim}{D}\,d\underset{\sim}{x} - d\underset{\sim}{x}\cdot \underset{\sim}{F}_e^{-T}\underset{\sim}{D}^*\underset{\sim}{F}_e^{-1}d\underset{\sim}{x} = d\underset{\sim}{x}\cdot(\underset{\sim}{D} - \underset{\sim}{F}_e^{-T}\underset{\sim}{D}^*\underset{\sim}{F}_e^{-1})d\underset{\sim}{x}.$$

This motivates the definition of plastic and elastic rate of deformation tensors by

$$\overset{(2)}{\underset{\sim}{D}}_p = \underset{\sim}{F}_e^{-T}\underset{\sim}{D}^*\underset{\sim}{F}_e^{-1}; \qquad (15.11)$$

$$\overset{(2)}{\underset{\sim}{D}}_e = \underset{\sim}{D} - \underset{\sim}{F}_e^{-T}\underset{\sim}{D}^*\underset{\sim}{F}_e^{-1}, \qquad (15.12)$$

the total stretching tensor beeing additively decomposed

$$\underset{\sim}{D} = \overset{(2)}{\underset{\sim}{D}}_e + \overset{(2)}{\underset{\sim}{D}}_p. \qquad (15.13)$$

An additive decomposition is also obtained if we take the material derivative of the total deformation gradient and use its multiplikative decomposition:

$$\begin{aligned}\underset{\sim}{L} &= \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} = (\underset{\sim}{F}_e\underset{\sim}{F}_p)^\cdot(\underset{\sim}{F}_e\underset{\sim}{F}_p)^{-1} = \dot{\underset{\sim}{F}}_e\underset{\sim}{F}_p\underset{\sim}{F}_p^{-1}\underset{\sim}{F}_e^{-1} + \underset{\sim}{F}_e\dot{\underset{\sim}{F}}_p\underset{\sim}{F}_p^{-1}\underset{\sim}{F}_e^{-1}\\ &= \dot{\underset{\sim}{F}}_e\underset{\sim}{F}_e^{-1} + \underset{\sim}{F}_e\underset{\approx}{L}^*\underset{\sim}{F}_e^{-1}.\end{aligned} \qquad (15.14)$$

Taking symmetric parts

$$D = \{ \dot{\underset{\sim}{E}}_e \underset{\sim}{E}_e^{-1} \} + \{ \underset{\sim}{E}_e \underset{\sim}{L}^* \underset{\sim}{E}_e^{-1} \} \qquad (15.15)$$

we identify formally the first and second term with

$$\overset{(1)}{\underset{\sim}{D}}_e = \{ \dot{\underset{\sim}{E}}_e \underset{\sim}{E}_e^{-1} \} \qquad (15.16)$$

$$\overset{(1)}{\underset{\sim}{D}}_p = \{ \underset{\sim}{E}_e \underset{\sim}{L}^* \underset{\sim}{E}_e^{-1} \}. \qquad (15.17)$$

and have again

$$\underset{\sim}{D} = \overset{(1)}{\underset{\sim}{D}}_e + \overset{(1)}{\underset{\sim}{D}}_p. \qquad (15.18)$$

$\overset{(1)}{\underset{\sim}{D}}_p$ as defined by (15.17) is not a suitable measure of the rate of plastic deformation, since it depends also on the plastic spin $\underset{\sim}{W}^*$.

$$\overset{(1)}{\underset{\sim}{D}}_p = \{ \underset{\sim}{E}_e (\underset{\sim}{D}^* + \underset{\sim}{W}^*) \underset{\sim}{E}_e^{-1} \} = \{ \underset{\sim}{E}_e \underset{\sim}{D}^* \underset{\sim}{E}_e^{-1} \} + \{ \underset{\sim}{E}_e \underset{\sim}{W}^* \underset{\sim}{E}_e^{-1} \} \qquad (15.19)$$

That is, $\overset{(1)}{\underset{\sim}{D}}_p$ can assume arbitrary values if $(\alpha \xi^{*2})^{\cdot} = 0$, since this condition only implies $\underset{\sim}{D}^* = \underset{\sim}{0}$, leaving $\underset{\sim}{W}^*$ unspecified.

In order to avoid these difficulties it is straightforward to define

$$\overset{(3)}{\underset{\sim}{D}}_p = \overset{(1)}{\underset{\sim}{D}}_p - \{ \underset{\sim}{E}_e \underset{\sim}{W}^* \underset{\sim}{E}_e^{-1} \} = \{ \underset{\sim}{E}_e \underset{\sim}{D}^* \underset{\sim}{E}_e^{-1} \}, \qquad (15.20)$$

$$\overset{(3)}{\underset{\sim}{D}}_e = \underset{\sim}{D} - \overset{(3)}{\underset{\sim}{D}}_p. \qquad (15.21)$$

These measures appear also in some natural manner if we use convected coordinates (see 15.2).

Since the rate of plastic deformation measures defined above all depend on $\underset{\sim}{D}^*$ the question arises whether we can not use $\mathbf{D}^*$ itself as plastic stretching tensor. However, $\mathbf{D}^*$ depends on the orientation of the intermediate configuration and is not physically acceptable unless the intermediate configuration will be specified. Lee [31] has used a special choice of the intermediate configuration such that the elastic transformation $\mathbf{F}_e$ becomes a pure deformation. With this special configuration (denoted by $\mathbf{F}_e'$, $\mathbf{F}_p'$) he defines

$$\mathbf{D}_p' = \{\mathbf{L}^{*\prime}\} = \{\dot{\mathbf{F}}_p'\mathbf{F}_p'^{-1}\}.$$

By use of the polar decomposition theorem

$$\mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e\,; \quad \mathbf{V}_e = \mathbf{V}_e^T\,; \qquad \mathbf{R}_e \text{ orthogonal};$$

it follows

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p = \mathbf{V}_e\mathbf{R}_e\mathbf{F}_p = \mathbf{F}_e'\mathbf{F}_p',$$

where $\mathbf{F}_e' = \mathbf{V}_e = \mathbf{V}_e^T\,; \quad \mathbf{F}_p' = \mathbf{R}_e\mathbf{F}_p.$

Hence

$$\begin{aligned}\mathbf{D}_p' &= \{\dot{\mathbf{F}}_p'\mathbf{F}_p'^{-1}\} = \{(\mathbf{R}_e\mathbf{F}_p)^{\cdot}(\mathbf{R}_e\mathbf{F}_p)^{-1}\}\\ &= \{(\dot{\mathbf{R}}_e\mathbf{F}_p + \mathbf{R}_e\dot{\mathbf{F}}_p)\mathbf{F}_p^{-1}\mathbf{R}_e^T\} = \{\dot{\mathbf{R}}_e\mathbf{R}_e^T\} + \{\mathbf{R}_e\mathbf{L}^*\mathbf{R}_e^T\}\\ &= \{\dot{\mathbf{R}}_e\mathbf{R}_e^T\} + \{\mathbf{R}_e\mathbf{W}^*\mathbf{R}_e^T\} + \{\mathbf{R}_e\mathbf{D}^*\mathbf{R}_e^T\} = \{\mathbf{R}_e\mathbf{D}^*\mathbf{R}_e^T\}, \qquad (15.22)\end{aligned}$$

since $\dot{\mathbf{R}}_e\mathbf{R}_e^T$ and $\mathbf{R}_e\mathbf{W}^*\mathbf{R}_e^T$ are skew.

It is also possible to select such specific intermediate configuration that the rate of plastic deformation defined by (15.17) becomes physically acceptable. Following Wang [37] let us take a particular intermediate configuration (denoted by $\bar{\mathbf{F}}_p$ and $\bar{\mathbf{F}}_e$) such that

$$\overline{\underset{\sim}{W}}^{*} = \frac{1}{2}\left(\dot{\bar{\underset{\sim}{F}}}_{p}\,\bar{\underset{\sim}{F}}_{p}^{-1} - \bar{\underset{\sim}{F}}_{p}^{-T}\,\dot{\bar{\underset{\sim}{F}}}_{p}^{T}\right) = \underset{\sim}{0} \tag{15.23}$$

and let us define

$$\bar{\underset{\sim}{D}}_{e} = \left\{ \dot{\bar{\underset{\sim}{F}}}_{e}\,\bar{\underset{\sim}{F}}_{e}^{-1} \right\} \;; \qquad \bar{\underset{\sim}{D}}_{p} = \left\{ \bar{\underset{\sim}{F}}_{e}\,\dot{\bar{\underset{\sim}{F}}}_{p}\,\bar{\underset{\sim}{F}}_{p}^{-1}\,\bar{\underset{\sim}{F}}_{e}^{-1} \right\} \tag{15.24}$$

We shall now show that

$$\bar{\underset{\sim}{D}}_{p} = \overset{(3)}{\underset{\sim}{D}}_{p}$$

where $\overset{(3)}{\underset{\sim}{D}}_{p}$ is defined by (15.20).

By substituting the following relation

$$\bar{\underset{\sim}{F}}_{e} = \underset{\sim}{F}_{e}\,\underset{\sim}{Q}^{*T}(t)\,; \quad \bar{\underset{\sim}{F}}_{p} = \underset{\sim}{Q}^{*}(t)\,\underset{\sim}{F}_{p}\;; \quad \left(\underset{\sim}{Q}^{*-1} = \underset{\sim}{Q}^{*T}\right) \tag{15.25}$$

into (15.23) it can be shown that

$$\underset{\sim}{Q}^{*\bullet} = -\underset{\sim}{Q}^{*}\,\underset{\sim}{W}^{*}(t) \tag{15.26}$$

where $\underset{\sim}{W}^{*}$ is the plastic spin defined by $(15.7)_3$ and $(15.5)_2$. This equation gives the variation of the transformation from arbitrary choosen intermediate configuration to that selected by Wang, i. e., if $\underset{\sim}{W}^{*}(t)$ is known one can solve equation (15.26) with respect to $\underset{\sim}{Q}^{*}$ and determine the variation in time of the orientation of Wang's intermediate configuration.

Now, substituting (15.25) into (15.24) and using (15.26) one eventually obtain the required result

$$\bar{\underset{\sim}{D}}_{e} = \overset{(1)}{\underset{\sim}{D}}_{e} + \left\{ \underset{\sim}{F}_{e}\,\underset{\sim}{W}^{*}\,\underset{\sim}{F}_{e}^{-1} \right\}; \quad \bar{\underset{\sim}{D}}_{p} = \overset{(1)}{\underset{\sim}{D}}_{p} - \left\{ \underset{\sim}{F}_{e}\,\underset{\sim}{W}^{*}\,\underset{\sim}{F}_{e}^{-1} \right\} = \overset{(3)}{\underset{\sim}{D}}_{p}\,.$$

Finally let us note that if the definition (15.24) is replaced by

$$\overline{\overline{\underset{\sim}{D}}}_p = \{ \overline{\underset{\sim}{F}}_e^{-T} \dot{\overline{\underset{\sim}{F}}}_p \overline{\underset{\sim}{F}}_p^{-1} \overline{\underset{\sim}{F}}_e^{-1} \}$$

then under constraints (15.23) we get

$$\overline{\overline{\underset{\sim}{D}}}_p = \overset{(2)}{\underset{\sim}{D}}_p .$$

It thus follows that within the same specification of the intermediate configuration we have still freedom in the definition of the rate of plastic deformation.

15.2. Description in Convected Coordinates

Using table (13.36), the material time derivatives of base vectors $\underset{\sim}{g}_i$, $\overset{*}{\underset{\sim}{g}}_i$, $\underset{\sim}{g}^i$, $\overset{*}{\underset{\sim}{g}}{}^i$ can be written down directly

$$\dot{\underset{\sim}{g}}_i = \dot{\underset{\sim}{F}} \overset{\circ}{\underset{\sim}{g}}_i = \dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1} \underset{\sim}{g}_i = \underset{\approx}{L} \underset{\sim}{g}_i \tag{15.27}$$

$$\dot{\underset{\sim}{g}}{}^i = (\underset{\sim}{F}^{-T})^{\cdot} \overset{\circ}{\underset{\sim}{g}}{}^i = (\underset{\sim}{F}^{-T})^{\cdot} \underset{\sim}{F}^T \underset{\sim}{g}^i = - \underset{\sim}{F}^{-T} \dot{\underset{\sim}{F}}{}^T \underset{\sim}{g}^i = - \underset{\approx}{L}^T \underset{\sim}{g}^i \tag{15.28}$$

$$\dot{\overset{*}{\underset{\sim}{g}}}_i = \dot{\underset{\sim}{F}}_p \overset{\circ}{\underset{\sim}{g}}_i = \dot{\underset{\sim}{F}}_p \underset{\sim}{F}_p^{-1} \overset{*}{\underset{\sim}{g}}_i = \underset{\approx}{L}^{*} \overset{*}{\underset{\sim}{g}}_i \tag{15.29}$$

$$\dot{\overset{*}{\underset{\sim}{g}}}{}^i = (\underset{\sim}{F}_p^{-T})^{\cdot} \overset{\circ}{\underset{\sim}{g}}{}^i = (\underset{\sim}{F}_p^{-T})^{\cdot} \underset{\sim}{F}_p^T \overset{*}{\underset{\sim}{g}}{}^i = - \underset{\approx}{L}^{*T} \overset{*}{\underset{\sim}{g}}{}^i \tag{15.30}$$

As consequences,

$$\underset{\approx}{L} = \dot{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}^i = (\underset{\sim}{g}_i \cdot \dot{\underset{\sim}{g}}_k) \, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k \tag{15.31}$$

$$\underset{\approx}{L}^{*} = \dot{\overset{*}{\underset{\sim}{g}}}_i \otimes \overset{*}{\underset{\sim}{g}}{}^i = (\overset{*}{\underset{\sim}{g}}_i \cdot \dot{\overset{*}{\underset{\sim}{g}}}_k) \, \overset{*}{\underset{\sim}{g}}{}^i \otimes \overset{*}{\underset{\sim}{g}}{}^k \tag{15.32}$$

where the covariant components are calculated from $\underset{\sim}{g}_i \cdot \underset{\approx}{L} \underset{\sim}{g}_k$ and $\overset{*}{\underset{\sim}{g}}_i \cdot \underset{\approx}{L}^{*} \overset{*}{\underset{\sim}{g}}_k$.

Stretching in $\mathcal{B}$ and $\mathcal{B}^*$ is now expressible by the metric changes $\dot{\underset{\sim}{g}}_{ik}$ and $\dot{\overset{*}{g}}_{ik}$,

$$\mathcal{D} = \{\underset{\sim}{L}\} = \frac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T) = \frac{1}{2}(\underset{\sim}{g}_i \cdot \dot{\underset{\sim}{g}}_k + \dot{\underset{\sim}{g}}_i \cdot \underset{\sim}{g}_k)\underset{\sim}{g}^i \otimes \underset{\sim}{g}^k = \frac{1}{2}\dot{g}_{ik}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^k$$

$$= \frac{1}{2} g^{ir}\dot{g}_{rk}\left[\frac{1}{2}(\underset{\sim}{g}_i \otimes \underset{\sim}{g}^k + \underset{\sim}{g}^k \otimes \underset{\sim}{g}_i)\right] = -\frac{1}{2}\dot{g}^{ik}\underset{\sim}{g}_i \otimes \underset{\sim}{g}_k \tag{15.33}$$

$$\mathcal{D}^* = \{\underset{\sim}{L}^*\} = \frac{1}{2}(\underset{\sim}{L}^* + \underset{\sim}{L}^{*T}) = \frac{1}{2}\dot{\overset{*}{g}}_{ik}\,\overset{*}{\underset{\sim}{g}}{}^i \otimes \overset{*}{\underset{\sim}{g}}{}^k \tag{15.34}$$

By use of (15.27)-(15.34) straightforward calculations would give the components in convected coordinates of the various measures of strain rate, defined in 15.1. Instead of doing it this way we prefer the opposite direction, showing first how strain rate components can be defined, using the convected coordinate system and then go back to the coordinate-free notation. All tensor components will be related to the base vectors $\underset{\sim}{g}_i$, $\underset{\sim}{g}^i$ of the convected coordinate system in the actual configuration.

According to (15.33) the total rate of deformation can be decomposed into

$$d_{ik} = \frac{1}{2}(g_{ik} - \overset{*}{g}_{ik})^{\boldsymbol{\cdot}} + \frac{1}{2}(\overset{*}{g}_{ik} - \overset{\circ}{g}_{ik})^{\boldsymbol{\cdot}} \tag{15.35}$$

The first and second term contain the metric in $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}^*$, $\mathcal{B}^\circ$, respectively and may therefore be used as measures of elastic and plastic rate of deformation

$$\underset{e}{\overset{(2)}{d}}{}_{ik} = \frac{1}{2}(g_{ik} - \overset{*}{g}_{ik})^{\boldsymbol{\cdot}} \tag{15.36}$$

$$\underset{p}{\overset{(2)}{d}}{}_{ik} = \frac{1}{2}(\overset{*}{g}_{ik} - \overset{\circ}{g}_{ik})^{\boldsymbol{\cdot}} = \frac{1}{2}\dot{\overset{*}{g}}_{ik} \tag{15.37}$$

With the same argument another decomposition follows from the contravariant components in (15.33)

$$-d^{ik} = \frac{1}{2}\dot{g}^{ik} = \frac{1}{2}(g^{ik} - \overset{*}{g}{}^{ik})^{\boldsymbol{\cdot}} + \frac{1}{2}(\overset{*}{g}{}^{ik} - \overset{\circ}{g}{}^{ik}) \tag{15.38}$$

$$\underset{e}{\overset{(4)}{d}}{}^{ik} = -\frac{1}{2}(g^{ik} - \overset{*}{g}{}^{ik})^{\boldsymbol{\cdot}} \tag{15.39}$$

$$\overset{(4)}{\underset{p}{d}}{}^{ik} = -\frac{1}{2}\left(\overset{*}{g}{}^{ik} - \overset{\circ}{g}{}^{ik}\right)^{\cdot} = -\frac{1}{2}\dot{\overset{*}{g}}{}^{ik} \quad {}^{*)} \tag{15.40}$$

The decomposition of the mixed components

$$d^{i}_{k} = \frac{1}{2} g^{ir}\dot{g}_{rk} \tag{15.41}$$

is more involved and has been worked out by Lehmann [38]. His starting point is the metric transformation matrix

$$q^{i}_{k} = \overset{\circ}{g}{}^{ir} g_{rk} = \overset{\circ}{g}{}^{ir}\overset{*}{g}_{rm}\overset{*}{g}{}^{ms} g_{sk} = \underset{p}{q}{}^{i}_{\cdot m}\,\underset{e}{q}{}^{m}_{k} \tag{15.42}$$

This is the component form of the multiplicative decomposition (cf. (14.9) and (14.11))

$$\underset{\sim}{B} = \underset{\sim}{B}{}_{p}'\,\underset{\sim}{B}{}_{e} \tag{15.43}$$

where

$$\underset{\sim}{B} = \overset{\circ}{g}{}^{ir} g_{rk}\,\frac{1}{2}\left(\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} + \underset{\sim}{g}^{k}\otimes\underset{\sim}{g}_{i}\right) = q^{i}_{k}\,\frac{1}{2}\left(\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} + \underset{\sim}{g}^{k}\otimes\underset{\sim}{g}_{i}\right) \tag{15.44}$$

$$\underset{\sim}{B}{}_{p}' = \overset{\circ}{g}{}^{ir}\overset{*}{g}_{rk}\,\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} = \underset{p}{q}{}^{i}_{\cdot k}\,\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} \tag{15.45}$$

$$\underset{\sim}{B}{}_{e} = \overset{*}{g}{}^{ir} g_{rk}\,\frac{1}{2}\left(\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} + \underset{\sim}{g}^{k}\otimes\underset{\sim}{g}_{i}\right) = \underset{e}{q}{}^{i}_{k}\,\frac{1}{2}\left(\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} + \underset{\sim}{g}^{k}\otimes\underset{\sim}{g}_{i}\right) \tag{15.46}$$

With

$$(\overset{-1}{q})^{i}_{k} = g^{ir}\overset{\circ}{g}_{rk} \tag{15.47}$$

---

*) Note, that $\overset{(4)}{\underset{p}{d}}{}_{ik} = -\frac{1}{2} g_{ir}(\overset{*}{g}{}^{rs})^{\cdot} g_{sk} = \frac{1}{2} g_{ir}\overset{*}{g}{}^{rm}\dot{\overset{*}{g}}_{ml}\overset{*}{g}{}^{ls} g_{sk} \neq \overset{(2)}{\underset{p}{d}}{}_{ik}$, the two decompositions beeing indeed different.

we have

$$
\begin{aligned}
2d^{i}_{k} &= (\overset{-1}{q})^{i}_{\cdot r}(q^{r}_{\cdot k})^{\cdot} = (q^{-1})^{i}_{\cdot r}\left(\underset{p}{\dot q}{}^{r}_{\cdot s}\,\underset{e}{q}{}^{s}_{\cdot k} + \underset{p}{q}{}^{r}_{\cdot s}\,\underset{e}{\dot q}{}^{s}_{\cdot k}\right)\\
&= (\overset{-1}{q})^{i}_{\cdot r}\,\underset{p}{\dot q}{}^{r}_{\cdot s}\,\underset{e}{q}{}^{s}_{\cdot k} + (\underset{e}{\overset{-1}{q}})^{i}_{\cdot s}\,\underset{e}{\dot q}{}^{s}_{\cdot k} \qquad (15.48)\\
&= 2\overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k} + 2\overset{(3)}{\underset{e}{d}}{}^{i}_{\cdot k}\,.
\end{aligned}
$$

Hence

$$\overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k} = \frac{1}{2}(\overset{-1}{q})^{i}_{\cdot r}\,\underset{p}{\dot q}{}^{r}_{\cdot s}\,q^{s}_{\cdot k} \qquad (15.49)$$

$$\overset{(3)}{\underset{e}{d}}{}^{i}_{\cdot k} = \frac{1}{2}(\underset{e}{\overset{-1}{q}})^{i}_{\cdot r}\,\underset{e}{\dot q}{}^{r}_{\cdot k} \qquad (15.50)$$

Expressed by the metric coefficients we have

$$\overset{(3)}{\underset{e}{d}}{}^{i}_{\cdot k} = \frac{1}{2}\,g^{ir}\overset{*}{g}_{rs}\,(\overset{*}{g}{}^{sl}g_{lk})^{\cdot} = \frac{1}{2}\left(g^{il}\dot g_{lk} + g^{ir}\overset{*}{g}_{rs}\,\dot{\overset{*}{g}}{}^{sl}g_{lk}\right) \qquad (15.51)$$

$$\overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k} = \frac{1}{2}\,g^{il}\dot{\overset{*}{g}}_{ls}\,\overset{*}{g}{}^{sr}g_{rk} \qquad (15.52)$$

The corresponding rate of deformation tensors are now defined by

$$
\begin{aligned}
&\underset{\sim}{\overset{(2)}{D}}{}_{p} = \overset{(2)}{\underset{p}{d}}{}_{ik}\,\underset{\sim}{g}^{i}\otimes\underset{\sim}{g}^{k} && \underset{\sim}{\overset{(2)}{D}}{}_{e} = \overset{(2)}{\underset{e}{d}}{}_{ik}\,\underset{\sim}{g}^{i}\otimes\underset{\sim}{g}^{k}\\
&\underset{\sim}{\overset{(3)}{D}}{}_{p} = \overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k}\,\frac{1}{2}\left(\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} + \underset{\sim}{g}^{k}\otimes\underset{\sim}{g}_{i}\right); && \underset{\sim}{\overset{(3)}{D}}{}_{e} = \overset{(3)}{\underset{e}{d}}{}^{i}_{\cdot k}\,\frac{1}{2}\left(\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}^{k} + \underset{\sim}{g}^{k}\otimes\underset{\sim}{g}_{i}\right) \qquad (15.53)\\
&\underset{\sim}{\overset{(4)}{D}}{}_{p} = \overset{(4)}{\underset{p}{d}}{}^{ik}\,\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}_{k}; && \underset{\sim}{\overset{(4)}{D}}{}_{e} = \overset{(4)}{\underset{e}{d}}{}^{ik}\,\underset{\sim}{g}_{i}\otimes\underset{\sim}{g}_{k}
\end{aligned}
$$

It is clear that all these measures are not effected by rigid body motions superimposed on the intermediate configuration because the metric remains invariant under such motions.

We still have to justify the notation in (15.53), since $\underset{\sim}{\overset{(2)}{D}}{}_{p}$ and $\underset{\sim}{\overset{(3)}{D}}{}_{p}$ are already defined in subchapter (15.1).

It is sufficient to do this for the plastic parts, the elastic parts beeing always $\mathbf{D}-\mathbf{D}_p$. We use (13.36) and (15.27)-(15.30).

$$\begin{aligned}\overset{(2)}{\underset{p}{d}}_{ik} &= \frac{1}{2}\,\dot{\overset{*}{g}}_{ik} = \frac{1}{2}\,(\overset{*}{\mathbf{g}}_i\cdot\overset{*}{\mathbf{g}}_k)^{\cdot} = \frac{1}{2}\,(\dot{\overset{*}{\mathbf{g}}}_i\cdot\overset{*}{\mathbf{g}}_k+\overset{*}{\mathbf{g}}_i\cdot\dot{\overset{*}{\mathbf{g}}}_k)\\ &= \frac{1}{2}\,(\mathbf{L}^*\overset{*}{\mathbf{g}}_i\cdot\overset{*}{\mathbf{g}}_k+\overset{*}{\mathbf{g}}_i\cdot\mathbf{L}^*\overset{*}{\mathbf{g}}_k) = \overset{*}{\mathbf{g}}_i\cdot\frac{1}{2}\,(\mathbf{L}^*+\mathbf{L}^{*T})\,\overset{*}{\mathbf{g}}_k\\ &= \overset{*}{\mathbf{g}}_i\cdot\mathbf{D}^*\overset{*}{\mathbf{g}}_k = \mathbf{F}_e^{-1}\mathbf{g}_i\cdot\mathbf{D}^*\mathbf{F}_e^{-1}\mathbf{g}_k\\ &= \mathbf{g}_i\cdot\mathbf{F}_e^{-T}\mathbf{D}^*\mathbf{F}_e^{-1}\mathbf{g}_k\end{aligned}\tag{15.54}$$

In fact, $\overset{(2)}{\mathbf{D}}_p = \overset{(2)}{\underset{p}{d}}_{ik}\,\mathbf{g}^i\otimes\mathbf{g}^k = \mathbf{F}_e^{-T}\mathbf{D}^*\mathbf{F}_e^{-1}.$ (15.55)

With the help of (15.54)

$$\begin{aligned}\overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k} &= g^{il}\,\frac{1}{2}\,\dot{\overset{*}{g}}_{ls}\,\overset{*}{g}{}^{sr}g_{rk} = g^{il}(\mathbf{g}_l\cdot\overset{(2)}{\mathbf{D}}_p\,\mathbf{g}_s)\,\overset{*}{g}{}^{sr}g_{rk}\\ &= \mathbf{g}^i\cdot\overset{(2)}{\mathbf{D}}_p\,\overset{*}{g}{}^{sr}g_{rk}\,\mathbf{g}_s\,.\end{aligned}$$

Now from (15.46) $\quad \overset{*}{g}{}^{rs}g_{rk}\,\mathbf{g}_s = \mathbf{B}_e\mathbf{g}_k = \mathbf{F}_e\mathbf{F}_e^T\mathbf{g}_k$

and $\quad \overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k} = \mathbf{g}^i\cdot\overset{(2)}{\mathbf{D}}_p\,\mathbf{F}_e\mathbf{F}_e^T\mathbf{g}_k = \mathbf{g}^i\cdot\mathbf{F}_e^{-T}\mathbf{D}^*\mathbf{F}_e^T\mathbf{g}_k\,.$

Hence $\quad \mathbf{F}_e^{-T}\mathbf{D}^*\mathbf{F}_e^T = \overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k}\,\mathbf{g}_i\otimes\mathbf{g}^k$

and $\quad \overset{(3)}{\mathbf{D}}_p = \overset{(3)}{\underset{p}{d}}{}^{i}_{\cdot k}\,\frac{1}{2}\,(\mathbf{g}_i\otimes\mathbf{g}^k+\mathbf{g}^k\otimes\mathbf{g}_i) = \{\mathbf{F}_e^{-T}\mathbf{D}^*\mathbf{F}_e^T\}.$ (15.56)

The coordinate-free representation of $\overset{(4)}{\mathbf{D}}_p$ is obtained from

$$\begin{aligned}\overset{(4)}{\underset{p}{d}}{}^{i}_{\cdot k} &= -\frac{1}{2}(\overset{*}{g}{}^{ik})^{\cdot} = -\frac{1}{2}(\dot{\overset{*}{\mathbf{g}}}{}^i\cdot\overset{*}{\mathbf{g}}{}^k+\overset{*}{\mathbf{g}}{}^i\cdot\dot{\overset{*}{\mathbf{g}}}{}^k) = \overset{*}{\mathbf{g}}{}^i\cdot\frac{1}{2}\,(\mathbf{L}^*+\mathbf{L}^{*T})\,\overset{*}{\mathbf{g}}{}^k\\ &= \overset{*}{\mathbf{g}}{}^i\cdot\mathbf{D}^*\overset{*}{\mathbf{g}}{}^k = \mathbf{F}_e^T\mathbf{g}^i\cdot\mathbf{D}^*\mathbf{F}_e^T\mathbf{g}^k = \mathbf{g}^i\cdot\mathbf{F}_e\mathbf{D}^*\mathbf{F}_e^T\mathbf{g}_k\end{aligned}$$

as

$$\overset{(4)}{\underset{\sim}{D}}_p = \underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^T . \tag{15.57}$$

Finally, we show how the elastic rate of deformation measures are related to time derivatives of elastic strain. Since

$$\underset{e}{\overset{(2)}{d}}_{ik} = \frac{1}{2} (g_{ik} - \overset{*}{g}_{ik})^{\cdot}$$

and, according to (14.13)

$$\frac{1}{2} (g_{ik} - \overset{*}{g}_{ik}) = \underset{\sim}{g}_i \cdot \underset{\sim}{\varepsilon}_e \underset{\sim}{g}_k \ ; \quad \underset{\sim}{\varepsilon}_e = \frac{1}{2} (\underset{\sim}{1} - \underset{\sim}{B}_e^{-1})$$

it follows

$$\underset{e}{\overset{(2)}{d}}_{ik} = (\underset{\sim}{g}_i \cdot \underset{\sim}{\varepsilon}_e \underset{\sim}{g}_k)^{\cdot} = \dot{\underset{\sim}{g}}_i \cdot \underset{\sim}{\varepsilon}_e \underset{\sim}{g}_k + \underset{\sim}{g}_i \cdot \dot{\underset{\sim}{\varepsilon}}_e \underset{\sim}{g}_k + \underset{\sim}{g}_i \cdot \underset{\sim}{\varepsilon}_e \dot{\underset{\sim}{g}}_k$$

$$= \underset{\sim}{g}_i \cdot (\dot{\underset{\sim}{\varepsilon}}_e + \underset{\sim}{L}^T \underset{\sim}{\varepsilon}_e + \underset{\sim}{\varepsilon}_e \underset{\sim}{L}) \underset{\sim}{g}_k \ ,$$

$$\overset{(2)}{\underset{\sim}{D}}_e = \dot{\underset{\sim}{\varepsilon}}_e + \underset{\sim}{L}^T \underset{\sim}{\varepsilon}_e + \underset{\sim}{\varepsilon}_e \underset{\sim}{L} = \overset{\Delta 2}{\underset{\sim}{\varepsilon}}_e = \frac{1}{2} (\underset{\sim}{1} - \underset{\sim}{B}_e^{-1})^{\Delta 2} \tag{15.58}$$

In the same way the equations for $\overset{(3)}{\underset{\sim}{D}}_e$ and $\overset{(4)}{\underset{\sim}{D}}_e$ are obtained

$$\overset{(3)}{\underset{\sim}{D}}_e = \frac{1}{2} \left\{ \underset{\sim}{B}_e^{-1} (\dot{\underset{\sim}{B}}_e - \underset{\sim}{L} \underset{\sim}{B}_e + \underset{\sim}{B}_e \underset{\sim}{L}) \right\} = \frac{1}{2} \left\{ \underset{\sim}{B}_e^{-1} \overset{\Delta 3}{\underset{\sim}{B}}_e \right\} \tag{15.59}$$

$$\overset{(4)}{\underset{\sim}{D}}_e = \frac{1}{2} (\underset{\sim}{B}_e - \underset{\sim}{1})^{\cdot} - \underset{\sim}{L} \frac{1}{2} (\underset{\sim}{B}_e - \underset{\sim}{1}) - \frac{1}{2} (\underset{\sim}{B}_e - \underset{\sim}{1}) \underset{\sim}{L}^T = \frac{1}{2} (\underset{\sim}{B}_e - \underset{\sim}{1})^{\Delta 4} \tag{15.60}$$

Thus, the elastic rate of deformation measures $\overset{(2)}{\underset{\sim}{D}}_e , \overset{(3)}{\underset{\sim}{D}}_e , \overset{(4)}{\underset{\sim}{D}}_e$ can be expressed by objective time derivatives of strain measures. It can be verified that $\overset{(1)}{\underset{\sim}{D}}_e$ admits the representation

$$\overset{(1)}{\underset{\sim}{D}}_e = \left\{ \dot{\underset{\sim}{F}}_e \underset{\sim}{F}_e^{-1} \right\} = \frac{1}{2} \left\{ \underset{\sim}{B}_e^{-1} (\dot{\underset{\sim}{B}}_e - \underset{\sim}{L}_e \underset{\sim}{B}_e + \underset{\sim}{B}_e \underset{\sim}{L}_e) \right\} ; \quad \underset{\sim}{L}_e = \dot{\underset{\sim}{F}}_e \underset{\sim}{F}_e^{-1} \tag{15.61}$$

which is formally like (15.59), however, $\underset{\sim}{L} = \dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1}$ replaced by $\underset{\sim}{L}_e = \dot{\underset{\sim}{F}}_e \underset{\sim}{F}_e^{-1}$.

15.3. Behaviour under Rigid Body Rotations

The transformation behaviour of material time derivatives of the deformation gradients is

$$\dot{\overline{\utilde{F}}} = (\utilde{Q}\utilde{F})^{\cdot} = \utilde{Q}^{\cdot}\utilde{F} + \utilde{Q}\dot{\utilde{F}} \tag{15.62}$$

$$\dot{\overline{\utilde{F}_e}} = (\utilde{Q}\utilde{F}_e\utilde{Q}^{*T})^{\cdot} = \utilde{Q}^{\cdot}\utilde{F}_e\utilde{Q}^{*T} + \utilde{Q}\dot{\utilde{F}}_e\utilde{Q}^{*T} + \utilde{Q}\utilde{F}_e\dot{\utilde{Q}}^{*T} \tag{15.63}$$

$$\dot{\overline{\utilde{F}_p}} = (\utilde{Q}^*\utilde{F}_p)^{\cdot} = \dot{\utilde{Q}}^*\utilde{F}_p + \utilde{Q}^*\dot{\utilde{F}}_p \tag{15.64}$$

Hence

$$\overline{\utilde{L}} = \dot{\overline{\utilde{F}}}\,\overline{\utilde{F}}^{-1} = (\utilde{Q}^{\cdot}\utilde{F} + \utilde{Q}\dot{\utilde{F}})\utilde{F}^{-1}\utilde{Q}^T = \utilde{Q}^{\cdot}\utilde{Q}^T + \utilde{Q}\utilde{L}\utilde{Q}^T \tag{15.65}$$

$$\overline{\utilde{L}^*} = \dot{\overline{\utilde{F}_p}}\,\overline{\utilde{F}_p^{-1}} = (\dot{\utilde{Q}}^*\utilde{F}_p + \utilde{Q}^*\dot{\utilde{F}}_p)\utilde{F}_p^{-1}\utilde{Q}^{*T} = \dot{\utilde{Q}}^*\utilde{Q}^{*T} + \utilde{Q}^*\utilde{L}^*\utilde{Q}^{*T} \tag{15.66}$$

$$\overline{\utilde{L}_e} = \dot{\overline{\utilde{F}_e}}\,\overline{\utilde{F}_e^{-1}} = (\utilde{Q}^{\cdot}\utilde{F}_e\utilde{Q}^{*T} + \utilde{Q}\dot{\utilde{F}}_e\utilde{Q}^{*T} + \utilde{Q}\utilde{F}_e\dot{\utilde{Q}}^{*T})\utilde{Q}^*\utilde{F}_e^{-1}\utilde{Q}^T \tag{15.67}$$

$$= \utilde{Q}^{\cdot}\utilde{Q}^T + \utilde{Q}\utilde{L}_e\utilde{Q}^T + \utilde{Q}\utilde{F}_e\dot{\utilde{Q}}^{*T}\utilde{Q}^*\utilde{F}_e^{-1}\utilde{Q}^T \tag{15.68}$$

$$\overline{\utilde{D}} = \{\overline{\utilde{L}}\} = \{\utilde{Q}\utilde{L}\utilde{Q}^T\} = \frac{1}{2}(\utilde{Q}\utilde{L}\utilde{Q}^T + \utilde{Q}\utilde{L}^T\utilde{Q}^T) = \utilde{Q}\utilde{D}\utilde{Q}^T \tag{15.69}$$

$$\overline{\utilde{D}^*} = \{\overline{\utilde{L}^*}\} = \utilde{Q}^*\utilde{D}^*\utilde{Q}^{*T} \tag{15.70}$$

where in the last two equations use has been made of the fact that $\utilde{Q}^{\cdot}\utilde{Q}^T$ and $\dot{\utilde{Q}}^*\utilde{Q}^{*T}$ are skew tensors.
From

$$\overline{\utilde{F}_e} = \utilde{Q}\utilde{F}_e\utilde{Q}^{*T} = \utilde{Q}\utilde{V}_e\utilde{R}_e\utilde{Q}^{*T} = \utilde{Q}\utilde{V}_e\utilde{Q}^T\utilde{Q}\utilde{R}_e\utilde{Q}^{*T} = \overline{\utilde{V}_e}\,\overline{\utilde{R}_e} \tag{15.71}$$

it can be seen that $\utilde{R}_e$ transforms like $\utilde{F}_e$, i. e.

$$\overline{\utilde{R}_e} = \utilde{Q}\utilde{R}_e\utilde{Q}^{*T}. \tag{15.72}$$

It is now easy to establish the corresponding relations for the various plastic and elastic rate of deformation tensors.

Apart of $\overset{(1)}{\underset{\sim}{D}}_p$ and $\overset{(1)}{\underset{\sim}{D}}_e$ which transform according to

$$\overline{\overset{(1)}{\underset{\sim}{D}}_p} = \{\overline{\underset{\sim}{F}_e \underset{\sim}{L}^* \underset{\sim}{F}_e^{-1}}\} = \{\underset{\sim}{Q}\underset{\sim}{F}_e \underset{\sim}{Q}^{*T}(\dot{\underset{\sim}{Q}}^* \underset{\sim}{Q}^{*T} + \underset{\sim}{Q}^* \underset{\sim}{L}^* \underset{\sim}{Q}^{*T})\underset{\sim}{Q}^* \underset{\sim}{F}_e^{-1} \underset{\sim}{Q}^T\}$$

$$= \{\underset{\sim}{Q}\underset{\sim}{F}_e \underset{\sim}{Q}^{*T}\dot{\underset{\sim}{Q}}^* \underset{\sim}{F}_e^{-1}\underset{\sim}{Q}^T + \underset{\sim}{Q}\underset{\sim}{F}_e \underset{\sim}{L}^* \underset{\sim}{F}_e^{-1}\underset{\sim}{Q}^T\}$$

$$= \underset{\sim}{Q}\overset{(1)}{\underset{\sim}{D}}_p \underset{\sim}{Q}^T + \underset{\sim}{Q}\{\underset{\sim}{F}_e \underset{\sim}{Q}^{*T}\dot{\underset{\sim}{Q}}^* \underset{\sim}{F}_e^{-1}\}\underset{\sim}{Q}^T \qquad (15.73)$$

$$\overline{\overset{(1)}{\underset{\sim}{D}}_e} = \overline{\underset{\sim}{D}} - \overline{\overset{(1)}{\underset{\sim}{D}}_p} = \underset{\sim}{Q}\overset{(1)}{\underset{\sim}{D}}_e \underset{\sim}{Q}^T - \underset{\sim}{Q}\{\underset{\sim}{F}_e \underset{\sim}{Q}^{*T}\dot{\underset{\sim}{Q}}^* \underset{\sim}{F}_e^{-1}\}\underset{\sim}{Q}^T \qquad (15.74)$$

all other measures are objective.
For example,

$$\overline{\overset{(3)}{\underset{\sim}{D}}_p} = \{\overline{\underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^{-1}}\} = \{\underset{\sim}{Q}\underset{\sim}{F}_e \underset{\sim}{Q}^{*T}\underset{\sim}{Q}^* \underset{\sim}{D}^* \underset{\sim}{Q}^{*T}\underset{\sim}{Q}^* \underset{\sim}{F}_e^{-1}\underset{\sim}{Q}^T\}$$

$$= \{\underset{\sim}{Q}\underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^{-1}\underset{\sim}{Q}^T\} = \underset{\sim}{Q}\{\underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^{-1}\}\underset{\sim}{Q}^T = \underset{\sim}{Q}\overset{(3)}{\underset{\sim}{D}}_p \underset{\sim}{Q}^T \qquad (15.75)$$

$$\overline{\underset{\sim}{D}'_p} = \overline{\underset{\sim}{R}}_e \overline{\underset{\sim}{D}^*}\,\overline{\underset{\sim}{R}}_e^T = \underset{\sim}{Q}\underset{\sim}{R}_e \underset{\sim}{Q}^{*T}\underset{\sim}{Q}^* \underset{\sim}{D}^* \underset{\sim}{Q}^{*T}\underset{\sim}{Q}^* \underset{\sim}{R}_e^T \underset{\sim}{Q}^T = \underset{\sim}{Q}\underset{\sim}{D}'_p \underset{\sim}{Q}^T \qquad (15.76)$$

The main results are summarized below.

Behaviour under transformation	Quantities
$\overline{\underset{\sim}{X}} = \underset{\sim}{Q}\underset{\sim}{X}\underset{\sim}{Q}^T$	$\overset{(2)}{\underset{\sim}{D}}_p, \overset{(3)}{\underset{\sim}{D}}_p, \overset{(4)}{\underset{\sim}{D}}_p, \underset{\sim}{D}'_p$ $\overset{(2)}{\underset{\sim}{D}}_e, \overset{(3)}{\underset{\sim}{D}}_e, \overset{(4)}{\underset{\sim}{D}}_e, \underset{\sim}{D}$
$\overline{\underset{\sim}{X}} = \underset{\sim}{Q}^*\underset{\sim}{X}\underset{\sim}{Q}^{*T}$	$\underset{\sim}{D}^*$
more complicated	$\dot{\underset{\sim}{F}}, \dot{\underset{\sim}{F}}_e, \dot{\underset{\sim}{F}}_p, \underset{\sim}{L}, \underset{\sim}{L}^*$

(15.77)

15.4. Summary and Comparison

For ease of comparison the results of this chapter are summarized in table (15.78). Exept for $\overset{(1)}{\underset{\sim}{D}}_p$ which behaves different all other measures of plastic rate of deformation have similar structure. They are based on $\underset{\sim}{D}^*$, the stretching of the intermediate configuration, and modified by the elastic transformation $\underset{\sim}{F}_e$. Interpretation in convected coordinates shows that $\overset{(2)}{\underset{\sim}{D}}_p$, $\overset{(3)}{\underset{\sim}{D}}_p$, $\overset{(4)}{\underset{\sim}{D}}_p$ are obtained by relating the covariant, contravariant and mixed components of $\underset{\sim}{D}^*$ to the corresponding convected base in the actual configuration. Lee's measure $\underset{\sim}{D}_p'$ can also be expressed by metric coefficients and their time derivatives. Since, however, the square root of $\underset{\sim}{B}_e = \underset{\sim}{V}_e^2$ is involved complicated irrational functions would result. Whereas vanishing change of arc length in the intermediate configuration, $(ds^{*2})^\cdot = 0$, implies vanishing plastic rate of deformation it does not follow from $(ds^2 - ds^{*2})^\cdot = 0$ that the elastic rates are zero. This will only be true for $\overset{(2)}{\underset{\sim}{D}}_e$ which thus has a special property.

The differences in all measures (always apart from $\overset{(1)}{\underset{\sim}{D}}_p$, $\overset{(1)}{\underset{\sim}{D}}_e$) disappear if the elastic deformation can be considered as infinitesimal. This condition will frequently be met when dealing with metal plasticity unless the hydrostatic pressure is very high. The principal stretches of the elastic transformation beeing small, $\underset{\sim}{F}_e$ can be written in the form

$$\underset{\sim}{F}_e = \underset{\sim}{V}_e \underset{\sim}{R}_e = (\underset{\sim}{1} + \underset{\sim}{\gamma})\underset{\sim}{R}_e \qquad |\underset{\sim}{\gamma}| = |\underset{\sim}{\gamma}^T| = \varepsilon \ll 1 \qquad (15.79)$$

Substituting (15.68) for $\underset{\sim}{F}_e$ and neglecting $\underset{\sim}{\gamma}$ compared to unity, results in

$$\overset{(2)}{\underset{\sim}{D}}_p \approx \overset{(3)}{\underset{\sim}{D}}_p \approx \overset{(4)}{\underset{\sim}{D}}_p \approx \underset{\sim}{D}_p' = \underset{\sim}{R}_e \underset{\sim}{D}^* \underset{\sim}{R}_e^T . \qquad (15.80)$$

	absolute notation	convected coordinates	relation to strain	comments	$\overline{\underset{\sim}{X}} = \underset{\sim}{Q}\underset{\sim}{X}\underset{\sim}{Q}^T$
$\overset{(1)}{\underset{\sim}{D}}_p$	$\{\underset{\sim}{F}_e \underset{\sim}{L}^* \underset{\sim}{F}_e^{-1}\}$	$(\overset{*}{\underset{\sim}{g}}^i \cdot \dot{\overset{*}{\underset{\sim}{g}}}_k)\{\underset{\sim}{g}_i \otimes \underset{\sim}{g}^k\}$			no
$\overset{(2)}{\underset{\sim}{D}}_p$	$\underset{\sim}{F}_e^{-T} \underset{\sim}{D}^* \underset{\sim}{F}_e^{-1}$	$\frac{1}{2} \dot{\overset{*}{g}}_{ik}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k$	$\frac{1}{2}(\underset{\sim}{B}_e^{-1} - \underset{\sim}{B}^{-1})^{\Delta 2} = \underset{\sim}{\varepsilon}_p^{\Delta 2}$	$\underset{\sim}{B}_e = \underset{\sim}{F}_e \underset{\sim}{F}_e^T;\ \underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T$	yes
$\overset{(3)}{\underset{\sim}{D}}_p$	$\{\underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^{-1}\} = \{\underset{\sim}{F}_e^{-T} \underset{\sim}{D}^* \underset{\sim}{F}_e^T\}$	$\frac{1}{2} \overset{*}{g}^{ir} \dot{\overset{*}{g}}_{rk} \{\underset{\sim}{g}_i \otimes \underset{\sim}{g}^k\}$	$\frac{1}{2}\{(\underset{\sim}{B}'_p)^{-1}(\underset{\sim}{B}'_p)^{\Delta 3}\}$	$\underset{\sim}{B}'_p = \underset{\sim}{F}_e \underset{\sim}{B}_p \underset{\sim}{F}_e^{-1};\ \underset{\sim}{B}_p = \underset{\sim}{F}_p \underset{\sim}{F}_p^T$	yes
$\overset{(4)}{\underset{\sim}{D}}_p$	$\underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^T$	$-\frac{1}{2} \dot{\overset{*}{g}}^{ik}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}_k$	$\frac{1}{2}(\underset{\sim}{B} - \underset{\sim}{B}_e)^{\Delta 4}$		yes
$\overset{(1)}{\underset{\sim}{D}}_e$	$\underset{\sim}{D} - \underset{\sim}{D}_p$.	$\underset{\sim}{D} - \underset{\sim}{D}_p$	$\frac{1}{2}\{\underset{\sim}{B}_e^{-1}(\dot{\underset{\sim}{B}}_e - \underset{\sim}{L}_e \underset{\sim}{B}_e + \underset{\sim}{B}_e \underset{\sim}{L}_e)\}$	$\underset{\sim}{L}_e = \dot{\underset{\sim}{F}}_e \underset{\sim}{F}_e^{-1}$	no
$\overset{(2)}{\underset{\sim}{D}}_e$			$\frac{1}{2}(\underset{\sim}{1} - \underset{\sim}{B}_e^{-1})^{\Delta 2} = \underset{\sim}{\varepsilon}_e^{\Delta 2}$	$(\ )^{\Delta 2} = (\ )^{\cdot} + \underset{\sim}{L}^T(\ ) + (\ )\underset{\sim}{L}$	yes
$\overset{(3)}{\underset{\sim}{D}}_e$			$\frac{1}{2}\{\underset{\sim}{B}_e^{-1} \underset{\sim}{B}_e^{\Delta 3}\}$	$(\ )^{\Delta 3} = (\ )^{\cdot} - \underset{\sim}{L}(\ ) + (\ )\underset{\sim}{L}$	yes
$\overset{(4)}{\underset{\sim}{D}}_e$			$\frac{1}{2}(\underset{\sim}{B}_e - \underset{\sim}{1})^{\Delta 4}$	$(\ )^{\Delta 4} = (\ )^{\cdot} - \underset{\sim}{L}(\ ) - (\ )\underset{\sim}{L}^T$	yes
$\underset{\sim}{D}^*$	$\{\underset{\sim}{L}^*\} = \{\dot{\underset{\sim}{F}}_p \underset{\sim}{F}_p^{-1}\}$	$\frac{1}{2} \dot{\overset{*}{g}}_{ik}\, \overset{*}{g}^i \otimes \overset{*}{\underset{\sim}{g}}^k = -\frac{1}{2} \dot{\overset{*}{g}}^{ik}\, \overset{*}{\underset{\sim}{g}}_i \otimes \overset{*}{\underset{\sim}{g}}_k$ $= \frac{1}{2} \overset{*}{g}^{ir} \dot{\overset{*}{g}}_{rk} \{\overset{*}{\underset{\sim}{g}}_i \otimes \overset{*}{\underset{\sim}{g}}^k\}$			no
$\underset{\sim}{D}$	$\{\underset{\sim}{L}^*\} = \{\dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1}\}$	$\frac{1}{2} \dot{g}_{ik}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^k = -\frac{1}{2} \dot{g}^{ik}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}_k$ $= \frac{1}{2} \dot{g}^{ir} g_{rk} \{\underset{\sim}{g}_i \otimes \underset{\sim}{g}^k\}$			yes

$\{\underset{\sim}{X}\}$ = symmetric part $= \frac{1}{2}(\underset{\sim}{X} + \underset{\sim}{X}^T)$

Table (15.78)

The procedure has to be applied with more care to $\overset{(1)}{\underset{\sim}{D}}_p$. Retaining the linear term in $\underset{\sim}{\gamma}$, we have

$$
\begin{aligned}
\overset{(1)}{\underset{\sim}{D}}_p &= \{ \underset{\sim}{F}_e \underset{\sim}{D}^* \underset{\sim}{F}_e^{-1} \} + \{ \underset{\sim}{F}_e \underset{\sim}{W}^* \underset{\sim}{F}_e^{-1} \} \\
&= \{ (\underset{\sim}{1}+\underset{\sim}{\gamma}) \underset{\sim}{R}_e \underset{\sim}{D}^* \underset{\sim}{R}_e^T (\underset{\sim}{1}+\underset{\sim}{\gamma})^{-1} \} + \{ (\underset{\sim}{1}+\underset{\sim}{\gamma}) \underset{\sim}{R}_e \underset{\sim}{W}^* \underset{\sim}{R}_e^T (\underset{\sim}{1}+\underset{\sim}{\gamma})^{-1} \} \\
&\simeq \{ (\underset{\sim}{1}+\underset{\sim}{\gamma}) \underset{\sim}{R}_e \underset{\sim}{D}^* \underset{\sim}{R}_e^T (\underset{\sim}{1}-\underset{\sim}{\gamma}) \} + \{ (\underset{\sim}{1}+\underset{\sim}{\gamma}) \underset{\sim}{R}_e \underset{\sim}{W}^* \underset{\sim}{R}_e^T (\underset{\sim}{1}-\underset{\sim}{\gamma}) \} \\
&\approx \underset{\sim}{R}_e \underset{\sim}{D}^* \underset{\sim}{R}_e^T + \underset{\sim}{\gamma} \underset{\sim}{R}_e \underset{\sim}{W}^* \underset{\sim}{R}_e^T - \underset{\sim}{R}_e \underset{\sim}{W}^* \underset{\sim}{R}_e^T \underset{\sim}{\gamma}
\end{aligned}
\qquad (15.81)
$$

The members with $\underset{\sim}{\gamma}$ are not necessarily small compared with $\underset{\sim}{R}_e \underset{\sim}{D}^* \underset{\sim}{R}_e^T$ since the plastic spin $\underset{\sim}{W}^*$ can be made arbitrarily large.

REFERENCES

[1] H.A. Buchdahl: The concepts of classical thermodynamics, Cambridge University Press 1966

[2] H.B. Callen: Thermodynamics, Ed. I. Wiley and Sons, New York, London 1960

[3] A.A. Vakulenko: A thermodynamik investigation of stress-strain relations in isotropic elastic-plastic solids, Dokl. A.N. SSSR, 126, 1959, 736-739 (in Russian)

[4] A.A. Vakulenko: Contributions to theory of elasticity and plasticity and theoretical problems of structural mechanics, Lectures presented 22-25 XII, 1954, Izd. A.N. SSSR (in Russian)

[5] J. Kestin: Thermodynamics in thermoplasticity, in "Thermoplasticity" Ed. "Ossolineum", Wroclaw, 1973, Lecture notes for Summer School in Jablonna (in Polish)

[6] J. Kestin and J.R. Rice: Paradoxes in the application of thermodynamics to strained solids, in "A Critical Review of Thermodynamics", Mono Book Corp. 1970

[7] Th. Lehmann: Some thermodynamic considerations of phenomenological theory of non-isothermal elastic-plastic deformations, Arch. Mech. Stos. 24 (1972), 975-984

[8] Th. Lehmann: Große elasto-plastiche Formänderungen, Mitteilungen aus dem Institut für Mechanik, Ruhr-Universität Bochum, 1 (1976)

[9] Th. Lehmann: Some aspects of non-isothermal large inelastic deformations, General Lecture presented at XIX. Polish Solid Mechanics Conference, Ruciane-Piaski, 7-16, Sept. 1977

[10] J. Gyarmati: Non-equilibrium thermodynamics, Field theory and variational principles, Springer, Berlin, New York 1970

[11] D.G. Edelen: On the existence of symmetry relations and dissipation potentials, Arch. Rational Mech. Anal. 51 (1973), 218

[12] D.G. Edelen: On the characterization of fluxes in nonlinear irreversible thermodynamics, Int. J. Engn. Sci. 12 (1974), 397-411

[13] J. Mandel: Plasticite classique et viscoplasticite, International Centre for Mechanical Sciences (CISM), Courses and Lectures No. 97, Udine 1971, Springer

[14] B. Halphen and Nguyen Quoc Son: Sur les materiaux standarts généralsés, Journal de Mécanique 14 (1975), No. 1

[15] B. Halphen: Sur le champ des vitesses en thermoplasticité finite, Int. J. Solids Structures 11 (1975), 947-960

[16] W. Prager: Non isothermal plastic deformation, Proc. Koninkl. Nederl. Akad. van Wetenschappen, Ser. B, 61 (1958), No. 4

[17] Z. Mróz and B. Raniecki: On the uniqueness problem in coupled thermoplasticity, Int. J. Eng. Sci. 14 (1976), 211-221

[18] B. Raniecki and A. Sawczuk: Thermal effects in plasticity, part I, Coupled theory, Zeitschr. Angew. Math. Mech. (ZAMM) 55 (1975), 331-341

[19] S.R. de Groot and P. Mazur: Non-equilibrium thermodynamics, Amsterdam 1962

[20] S. Wiśniewski, B. Staniszewski and R. Szymanik: Thermodynamics of nonequilibrium processes, PWN and D. Reidel Publ. Comp. Dordrecht, Holland, 1976

[21] J. Batailie and J. Kestin: L'interprétation physique de la thermodynamique rationnelle, Journal de Mécanique 14 (1975), No. 2

[22] A.C. Eringen: Nonlinear theory of continuous media, McGraw-Hill Book Comp., New York 1962

[23] I. Mueller: Entropy, absolute temperature and coldness in thermodynamics, Course held in CISM, Udine, 1971, Lecture Notes, Springer, Wien, New York

[24] B. Noble and M.J. Sewell: On dual extremum principles in applied mathematics, J. Inst. Math. Applies 9 (1972), 123-193

[25] M.B. Bever, D.L. Holt and A.L. Titchener: The stored energy of cold work, in:Progress in Material Sciences 17, Pergamon Press, 1973

[26] J.F. Nye: Physical properties of crystals, Clarendon Press Oxford 1957

[27] Robert E. Hill: Physical metallurgy principles, D. van Nostrand Comp. 1964

[28] S. Nemat-Nasser: On nonequilibrium thermodynamics of continua, Mechanics Today 2 (1975)

[29] L.I. Sedov: Mechanics of continuous medium 2, Nauka Moscov 1970 (in Russian, there is also English translation)

[30] C. Eckart: Theory of elasticity and anelasticity, Phys. Rev. 73 (1948), 373

[31] E.H. Lee, Elastic-plastic deformation at finite strains, J. Appl. Mech. 36 (1969), 1-6

[32] Th. Lehmann: Some aspects of non-isothermic large inelastic deformations, Solid Mech. Arch. 3 (1978)

[33] F. Sidoroff: The geometrical concept of intermediate configuration and elastic-plastic finite strain, Arch. Mech. 25 (1973), 299-309

[34] A.E. Green and P.M. Naghdi: Some remarks on elastic-plastic deformation at finite strain, Int. J. Engng. Sci. 9 (1971), 1219-1229

[35] A.E. Green and P.M. Naghdi: Thermodynamic development of elastic-plastic continua, in: Proc. IUTAM Symp. on Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluid, Springer 1966

[36] A.E. Green and P.M. Naghdi: General theory of an elastic-plastic continuum, Arch. Rat. Mech. Anal. 18 (1965), 251-281

[37] Y.S. Wang: A simplified theory of the constitutive equations of metal plasticity at finite deformation, J. Appl. Mech. 40 (1973), 941-947

[38] Th. Lehmann: On large elastic-plastic deformations, in: Papers contr. to the Intern. Symp. on Foundations of Plasticity, Warsaw 1972, Noordhoff 1973, 571-585

APPENDIX A

1. Legendre Transformation [24]

Let be given a continuous scalar function

$$Y = Y(x_i, u_\alpha) \tag{A1}$$

of an n-vector $x = \{x_i\}$, $i = 1, \ldots n$ and an m-vector $u = \{u_\alpha\}$ $\alpha = 1, \cdots m$. Suppose that Y has continuous first derivarives. Let there be two more vectors: n-vector $y = \{y_i\}$ and m-vector $v = \{v_\alpha\}$ which are related with x and u by the following equations

$$y_i = \frac{\partial Y}{\partial x_i} \quad ; \quad v_\alpha = \frac{\partial Y}{\partial u_\alpha}. \tag{A2}$$

Equation $(A2)_1$ define a transformation from the $\{x_i, u_\alpha\}$-space to the $\{y_i\}$ -space which we suppose to be single-valued (for each u_α) and to have an inverse expressing each x_i in terms of y_i and u_α. Therefore we assume that

$$det \left[\frac{\partial^2 Y}{\partial x_i \partial x_j} \right] \neq 0. \tag{A3}$$

Now, consider the following problem: to show that there exists a scalar function $X = X(x_i, u_\alpha)$ in terms of which just mentioned inverse transformation is expressible as $x_i = \frac{\partial X}{\partial y_i}$. The solution of this problem is most straightforward by using the technique of Legendre transformations.

The Legendre transform X of function Y is defined as

$$X(y_i, u_\alpha) = \left[y_j x_j - Y \right]_{x_i = x_i(y_i, u_\alpha)} \tag{A4}$$

where $x_i = x_i(y_i, u_\alpha)$ is the inverse of $(A2)_1$.

By differentiating (A4) with respect to y_i and by inserting (A2) into obtained result we find that the function $X(y_i, u_\alpha)$ is the solution of the problem:

$$\frac{\partial X}{\partial y_i} = \delta_{ij} x_j + y_j \frac{\partial x_j}{\partial y_i} - \frac{\partial Y}{\partial x_j} \frac{\partial x_j}{\partial y_i} = x_i \tag{A5}$$

Note that

$$\frac{\partial X}{\partial u_\alpha} = y_j \frac{\partial x_j}{\partial u_\alpha} - \frac{\partial Y}{\partial u_\alpha} - \frac{\partial Y}{\partial x_i} \frac{\partial x_i}{\partial u_\alpha} = - v_\alpha \tag{A6}$$

on account of (A2).

Whenever (A3) holds the development can be given in the reverse order, from (A6-A5) with given $X(y_i, u_\alpha)$ back to (A2) with Y defined by

$$Y = [\,y_j x_j - X\,]_{y_i = y_i(x_i, u_\alpha)} .$$

A Legendre dual transformation is therefore completely symmetrical with respect to its two "active sets of variables" x_i and y_i . The function X and Y are sometimes said to be <u>dual</u> of each other.

One can provide the following geometrical interpretation of a Legendre transformation: Let u_α be fixed, $u_\alpha = u_\alpha^\circ$. Then the locus of points (in $n+1$ dimensional space $\{Y, x_i\}$) satisfying (A1) is the hyper-surface S . The hyper-surface S (Fig. A1) may be equally well represented as the envelope of the tangent hyper-planes. The family of tangent hyper-planes may be characterized by giving the intercept $-X$ of a hyper-plane along the Y-axis as a function of the slopes y_i (cf. $(A2)_1$).

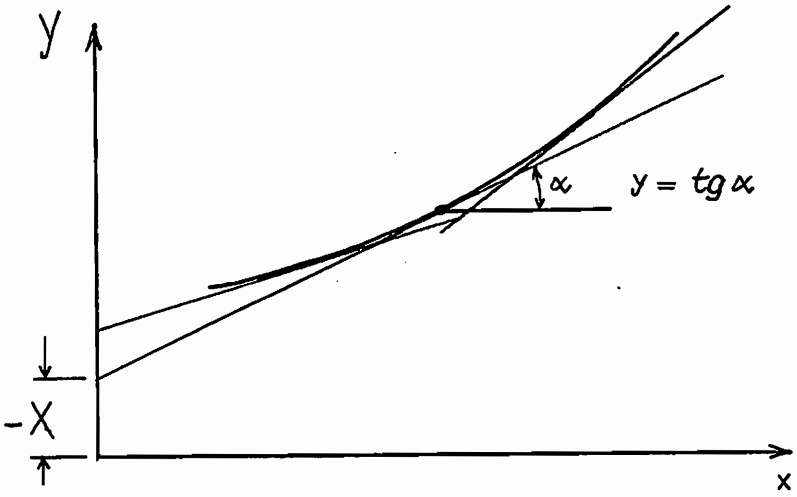

Fig. A1

2. Legendre Transformation when Function y is Homogeneous of Order One

Let the function (A1) be nonnegative and homogeneous of order one with respect to $\{x_i\}$ in the sense that

$$y\,(\tau x_i,\, u_\alpha) = \tau\, y\,(x_i, u_\alpha) \tag{A7}$$

for arbitrary $\tau > 0$ and every $\{u_\alpha\}$. Suppose also that $y = 0$ only if $x_i = 0$ and that y as a function of $\{x_i\}$ is of the class C^2 in the domain of determination excluding the point $\{x_i\} = \{0\}$. The Euler theorem then asserts that such a function has the property

$$\frac{\partial y}{\partial x_i}\, x_i = y\,(x_i, u_\alpha). \tag{A8}$$

The transformation $(A2)_1$ is now not a single valued since y_i is not uniquely determined at $\{x_i\} = \{0\}$. Moreover, the condition (A3) is not satisfied. Indeed, the differentiation of (A8) with respect to x_j yields

$$\frac{\partial^2 y}{\partial x_i \partial x_j} x_i = 0 \tag{A9}$$

The above set of equations should be satisfied for arbitrary vector $\{x_i\}$. Therefore

$$\det \frac{\partial^2 y}{\partial x_i \partial x_j} = 0 \tag{A10}$$

for all $\{x_i\} \neq \{0\}$ and the problem of finding the inverse relations with respect to that of $(A2)_1$ is more involved [42]. To find the inverse relations let us assume for simplicity that the rank of the matrix

$$\frac{\partial^2 y}{\partial x_i \partial x_j}$$

is $n-1$. Then from (A10) it follows that only $n-1$ functions determined by $(A2)_1$ are independent i. e., there exists a function we denote by F_1 such that

$$F_1(y_i, u_\alpha) = 0 \quad \text{if} \quad x_i \neq 0 \tag{A11}$$

Now consider the set of homogeneous equations for unknown $\{z_i\}$

$$\frac{\partial^2 y}{\partial x_i \partial x_j} z_j = 0. \tag{A12}$$

The general solution of this set can be written in the form

$$z_j = \Lambda\, z_j^*(x_i, u_\alpha)$$

where Λ is arbitrary multiplier and $z_j^*(x_i, u_\alpha)$ is any non-zero vector which satisfies (A12). This vector can be expressed as

$$z_j^* = \left.\frac{\partial F_1}{\partial y_j}\right|_{y_j = y_j(x_i, u_\alpha)}$$

since $F_1[y_j(x_i, u_\alpha), u_\alpha] = 0$ and

$$\frac{\partial F_1}{\partial y_j}\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial F_1}{\partial y_j}\frac{\partial y_j}{\partial x_i} = 0.$$

Thus, the general solution of (A12) can be presented in the form

$$z_j = \Lambda \frac{\partial F_1}{\partial y_j}.$$

Comparing (A9) with (A12) we see that for every $\{y_i\} \in \{F_1 = 0\}$ there correspond infinite number of $\{x_i\}$ which are determined by the formula

$$x_i = \Lambda \frac{\partial F_1}{\partial y_i}$$

and for every $\{y_i\} \in \{F_1 = 0\}$ the zero's vector $\{x_i\}$ is assigned

$$x_i = 0.$$

Conversely, suppose that x_i is defined by (A13) and F_1 is prescribed. Let $y_i = y_i(x_i, u_\alpha)$ be functions such that equation

$$F_1[y_j(x_i, u_\alpha), u_\alpha] = 0$$

is satisfied identically. Then one can show that

$$y_i = \frac{\partial y}{\partial x_i}$$

where y is homogeneous function of order one with respect to x_i.

Further specification of inverse relation can proceed as follows. For given u_α the locus of points (in the space $\{y_i\}$) satisfying the eq. $(A11)_1$ represents a surface. If this surface is regular (i. e. it has unique tangent plane at every point) then the derivatives $\partial F_1/\partial y_i$ can not all vanish at any point of a surface $F_1 = 0$ i. e. there exists such F_1 that

$$\frac{\partial F_1}{\partial y_i} = 0 \quad \text{while} \quad F_1 = 0.$$

Now, since $Y(x_i) = y_i x_i$ is positive except the point $x_i = 0$ at which Y vanishes, the pair (Λ, F_1) should satisfy the condition

$$Y = \Lambda \frac{\partial F_1}{\partial y_i} y_i \geqslant 0 \qquad \text{when} \qquad F_1 = 0$$

The product $y_i\, \partial F_1/\partial y_i$ can not vanish because otherwise there could exists $\Lambda \neq 0$ such that Y would be equal to zero at $x_i \neq 0$. To satisfy the above inequality, by convention, the sign of function F_1 is taken so that

$$\Lambda \geqslant 0 \quad \text{and} \quad \frac{\partial F_1}{\partial y_i} y_i > 0 \qquad \text{when} \qquad F_1 = 0.$$

Thus, when $F_1 = 0$ x_i can vanish only if $\Lambda = 0$, and extended form of the inverse relation is the following

$$\begin{aligned} x_i &= \Lambda \frac{\partial F_1}{\partial y_i} \qquad \text{if} \qquad F_1 = 0 \quad \text{and} \quad \Lambda \geqslant 0 \\ x_i &= 0 \qquad\qquad \text{if} \qquad F_1 \neq 0. \end{aligned} \qquad (A13)$$

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